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The Time-Reversal Symmetry vs. the Technical Condition of Proper Operation of the Asynchronous Flows

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Abstract

The asynchronous flows are generated by Boolean functions $\Phi : \{0,1\}^n \to \{0,1\}^n$ whose coordinates $\Phi_i, i = \overline{1,n}$ are iterated independently on each other. The order and the time instants of these iterations are not known. The flows are the models of the asynchronous circuits from the digital electronics. Time-reversal symmetry is one of the fundamental symmetries discussed in natural science and our main purpose is to adapt this concept, by analogy with mechanics, to the asynchronous flows. The technical condition of proper operation, also known as race-freedom, is a special case of work in asynchronicity which 'softens' the non-determinism of the models. We prove that when it is fulfilled, the flow behaves like a dynamical system. Finally, we relate the time-reversal symmetry and the fulfillement of the technical condition of proper operation.

1. Introduction

We denote in the following with $\mathbf{B}$ the Boolean algebra with two elements, i.e. the set $\{0,1\}$ endowed with the complement $'\ -\ '$, the intersection $'\ \cdot\ '$, the union $'\cup'$, and the modulo 2 sum $'\oplus'$.

The asynchronous flows describe the behavior of the digital devices from electronics. They are generated by Boolean functions $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ that iterate their coordinates $\Phi_1, ..., \Phi_n$ independently on each other.

In order to state the problem, let us consider Figure 1 where the state portrait of the function $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$,

$$\forall(\mu_1,\mu_2) \in \mathbf{B}^2, \Phi(\mu_1,\mu_2) = (\overline{\mu_1} \cup \mu_1 \cdot \overline{\mu_2}, \overline{\mu_1} \cup \mu_1 \cdot \mu_2)$$

was drawn. We have underlined the coordinates, so-called excited, or enabled, or unstable, for which $\Phi_i(\mu_1,\mu_2) \neq \mu_i, i = \overline{1,2}$. These coordinates are about to change their values, for example $(\underline{1},1)$ shows that $\Phi_1(1,1) = 0$; on the other hand the non-underlined coordinates

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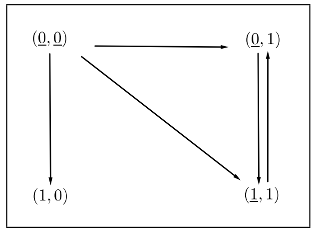

Figure 1. Example of state portrait.

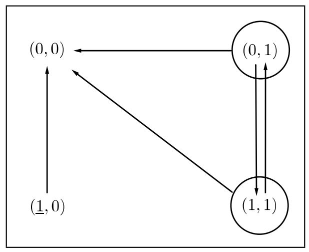

Figure 2. A time-reversal symmetry of the function Φ from Figure 1 and another function Ψ does not exist.

keep their values, for example $\Phi_2(1,1) = 1$. The arrows show the increase of time. Time is discrete in this paper, and the request is that the sets of the time instants when the coordinate functions Φ_1, Φ_2 are computed are infinite. If the flow is in $(0,0)$, it can go in three different directions, since both coordinates are excited, depending on which coordinate changes first: if $\Phi_1(0,0)$ is computed first, the next value of the flow is $(1,0)$, if $\Phi_2(0,0)$ is computed first, the next value is $(0,1)$ and if $\Phi_1(0,0), \Phi_2(0,0)$ are computed at the same time, the next value is $(1,1)$. If the flow is in $(1,0)$, it rests there indefinitely long, since $\Phi(1,0) = (1,0)$. And if the flow is in $(1,1)$ or $(0,1)$, it switches infinitely many times between these points.

Time-reversal symmetry is one of the fundamental symmetries discussed in natural science. Consequently, it arises in many physically motivated dynamical systems, in particular in classical and quantum mechanics. In the case of the asynchronous flows, time reversal symmetry means the inversion of the arrows in the state portraits. Figure 2 shows that the function Φ from Figure 1 is symmetrical with no other function Ψ. The definition of Ψ is possible in $(1,0)$ and $(0,0)$, but in $(1,1)$ and $(0,1)$ it is impossible, and we have pointed this out with the circles surrounding the two points. In fact, two arrows should start from $(1,1)$ in Figure 2; if we underline a coordinate, then one arrow exists, and if we underline two coordinates, then three arrows exist, making the definition of Ψ impossible.

Time-reversal symmetry exists of the function Φ from Figure 3 a) and the function Ψ from Figure 3 b). The state portraits of Φ, Ψ create the impression that the symmetrical flows run in opposite senses of time.

The concept of time-reversal symmetry is present in literature in the context of the (real, usual) dynamical systems and the main purpose of this paper is to adapt it for the asynchronous flows.

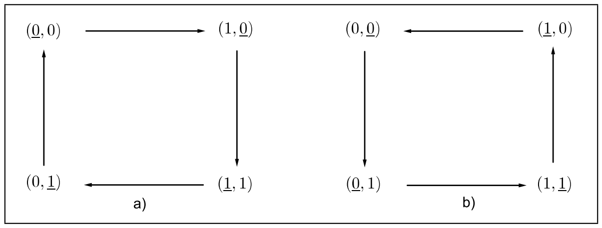

Figure 3. Time-reversal symmetry exists of the function Φ from a) and the function Ψ from b).

The model represented by the asynchronous flows presumes that neither the order of the iterations of $\Phi_i, i = \overline{1,n}$, nor the time instants when they happen are known, due to the uncertainties introduced by technology, temperature variations and voltage supply variations. These uncertainties make the model be non-deterministic, but in the special case of so-called 'race-freedom' (when we also say that the technical condition of proper operation is satisfied), it works 'deterministic-like' in the sense that we do not know exactly when the transitions from one state to another state happen, but we know that they happen sometime, see Figure 3 a) and Figure 3 b). The secret of this deterministic-like behavior is given by the request that each state has at most one exited coordinate. Relating time-reversal symmetry to the technical condition of proper operation is treated with attention in the paper.

The bibliography consists in the survey [1] on the time-reversal symmetry of the dynamical systems that generated analogies, together with works of ours in asynchronous flows.

2. Preliminaries

Definition 1. *For $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ and $\lambda \in \mathbf{B}^n$, we define the function $\Phi^\lambda : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ by* $\forall \mu \in \mathbf{B}^n, \forall i \in \{1, ..., n\}$,

$$\Phi_i^\lambda(\mu) = \begin{cases} \mu_i, if\ \lambda_i = 0, \\ \Phi_i(\mu), if\ \lambda_i = 1. \end{cases}$$

Definition 2. *The sequence $\alpha : \{0, 1, 2, ...\} \longrightarrow \mathbf{B}^n$, whose terms are denoted in general with α^k (instead of $\alpha(k)$), is called **progressive** if $\forall i \in \{1, ..., n\}$, the set $\{k|k \in \{0, 1, 2, ...\}, \alpha_i^k = 1\}$ is infinite. The set of the progressive sequences is denoted by $\widehat{\Pi}_n$.*

Definition 3. *The (**asynchronous**) **flow** $\widehat{\Phi}^\alpha(\mu, \cdot) : \{-1, 0, 1, ...\} \longrightarrow \mathbf{B}^n$ is defined by* $\forall k \geq -1$,

$$\widehat{\Phi}^\alpha(\mu, k) = \begin{cases} \mu,\ if\, k = -1, \\ \Phi^{\alpha^k}(\widehat{\Phi}^\alpha(\mu, k-1)),\ if\, k \geq 0. \end{cases}$$

*Φ is called the **generator function**, and μ is called the **initial (value of the) state**.*

Remark 4. *Here are the explanations related with the previous definitions. Unlike Φ that is computed on all its coordinates (at the same time), Φ^λ is computed on these coordinates* *only where $\lambda_i = 1$. $\widehat{\Phi}^{\alpha}(\mu, \cdot)$ represents the evolution of a state function starting from μ, that is given by the iterations of Φ_i, made independently on each other, at time instants and in an order indicated by the terms of α. The fact that α is progressive shows that any coordinate i is computed infinitely many times. And the fact that the processes that are modelled by these flows are influenced by unspecified parameters (such as technology, temperature, voltage supply) included indirectly in the model by α, is handled under the form: we are interested in special classes of functions Φ so that we can study the properties of $\widehat{\Phi}^{\alpha}(\mu, \cdot)$ that hold for all $\mu \in \mathbf{B}^n$ and all $\alpha \in \widehat{\Pi}_n$.*

Notation 5. *We denote by $\varepsilon^i \in \mathbf{B}^n$ the tuple $\varepsilon^i = (0, ..., \underset{i}{1}, ..., 0), i \in \{1, ..., n\}$.*

Remark 6. *$\mathbf{B}^n$ is a linear space over the field $\mathbf{B}$; the sum of the vectors is made coordinatewise $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n$,*

$$(\mu_1, ..., \mu_n) \oplus (\mu'_1, ..., \mu'_n) = (\mu_1 \oplus \mu'_1, ..., \mu_n \oplus \mu'_n)$$

and the multiplication with scalars from $\mathbf{B}$ is made coordinatewise too. ε^i are the vectors of the canonical basis of $\mathbf{B}^n$. Note that the sum $\mu \oplus \mu'$ shows which are the coordinates of μ, μ' that differ ($\mu_i \oplus \mu'_i = 1$) and which are the coordinates of μ, μ' that are equal ($\mu_i \oplus \mu'_i = 0$).

3. Predecessors and Successors

Definition 7. *Let $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ and $\mu \in \mathbf{B}^n$. Then $\mu' \in \mathbf{B}^n$ is called a **predecessor** of μ (via Φ) if $\lambda \in \mathbf{B}^n$ exists such that $\Phi^{\lambda}(\mu') = \mu$. The set of the predecessors of μ is denoted with μ^-, thus*

$$\mu^- = \{\mu' | \mu' \in \mathbf{B}^n, \exists \lambda \in \mathbf{B}^n, \Phi^{\lambda}(\mu') = \mu\}.$$

Definition 8. *The point $\mu' \in \mathbf{B}^n$ is called a **successor** of μ (via Φ) if $\lambda \in \mathbf{B}^n$ exists such that $\Phi^{\lambda}(\mu) = \mu'$. The set of the successors of μ is denoted by μ^+, in other words*

$$\mu^+ = \{\mu' | \mu' \in \mathbf{B}^n, \exists \lambda \in \mathbf{B}^n, \Phi^{\lambda}(\mu) = \mu'\}.$$

Remark 9. *For any Φ and μ, μ^- is non-empty, since $\mu \in \mu^-$ (we have $\Phi^{(0,...,0)}(\mu) = \mu$). Obviously, $\mu \in \mu^+$ and μ^+ is non-empty too.*

Example 10. *In Figure 4, we have $(1,1,1)^- = \{(1,1,0), (1,0,1), (0,1,1), (1,1,1)\}$ and $(1,1,1)^+ = \{(1,1,0), (0,1,1), (0,1,0), (1,1,1)\}$.*

Theorem 11. *Let $\mu \in \mathbf{B}^n$. The statements*

$$\Phi(\mu) = \mu, \tag{1}$$

$$\mu^+ = \{\mu\} \tag{2}$$

are equivalent; for any $p \in \{1, ..., n\}$ and $i_1 \in \{1, ..., n\}, ..., i_p \in \{1, ..., n\}$ distinct, the statements

$$\Phi(\mu) = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}, \tag{3}$$

$$\mu^+ = \{\mu \oplus \lambda_{i_1} \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p} \cdot \varepsilon^{i_p} | \lambda \in \mathbf{B}^n\} \tag{4}$$

are also equivalent.

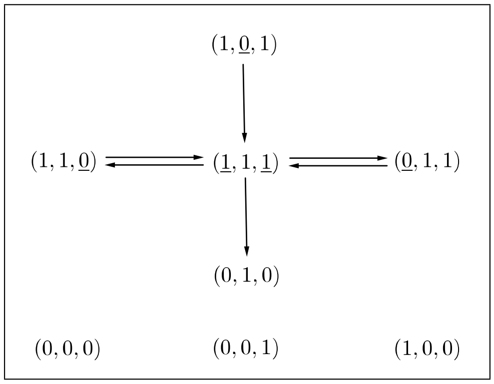

Figure 4. $(1,1,1)^{-} = \{(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$ and $(1,1,1)^{+} = \{(1,1,0),(0,1,1),(0,1,0),(1,1,1)\}$.

Proof. (1)$\Longrightarrow$(2) Since $\forall\lambda \in \mathbf{B}^n, \Phi^{\lambda}(\mu) = \mu$, we get $\mu^{+} = \{\Phi^{\lambda}(\mu)|\lambda \in \mathbf{B}^n\} = \{\mu\}$.

(2)$\Longrightarrow$(1) If $\{\Phi^{\lambda}(\mu)|\lambda \in \mathbf{B}^n\} = \{\mu\}$, then for $\lambda = (1,...,1) \in \mathbf{B}^n$ we infer $\Phi^{\lambda}(\mu) = \mu = \Phi(\mu)$.

(3)$\Longrightarrow$(4) Let $p \in \{1,...,n\}$ and $i_1 \in \{1,...,n\}$,..., $i_p \in \{1,...,n\}$ distinct. We compute the values of $\Phi_i^{\lambda}(\mu)$, where $i \in \{1,...,n\}, \lambda \in \mathbf{B}^n$; we obtain:

$$\Phi_i^{\lambda}(\mu) = \begin{cases} \mu_i, \text{ if } \lambda_i = 0, \\ \mu_i, \text{ if } \lambda_i = 1, i \notin \{i_1,...,i_p\}, \\ \mu_i \oplus 1, \text{ if } \lambda_i = 1, i \in \{i_1,...,i_p\}, \end{cases}$$

thus

$$\Phi^{\lambda}(\mu) = \mu \oplus \lambda_{i_1} \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p} \cdot \varepsilon^{i_p}.$$

We infer

$$\mu^{+} = \{\Phi^{\lambda}(\mu)|\lambda \in \mathbf{B}^n\} = \{\mu \oplus \lambda_{i_1} \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p} \cdot \varepsilon^{i_p}|\lambda \in \mathbf{B}^n\}.$$

(4)$\Longrightarrow$(3) We suppose against all reason that (3) is false and we have two possibilities.

Case $j \in \{1,...,n\} \setminus \{i_1,...,i_p\}$ exists with $\Phi_j(\mu) = \mu_j \oplus 1$

Then $\mu \oplus \varepsilon^j \in \{\Phi^{\lambda}(\mu)|\lambda \in \mathbf{B}^n\} \setminus \{\mu \oplus \lambda_{i_1} \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p} \cdot \varepsilon^{i_p}|\lambda \in \mathbf{B}^n\}$, contradiction with (4).

Case $j \in \{i_1,...,i_p\}$ exists with $\Phi_j(\mu) = \mu_j$

In this situation $\mu \oplus \varepsilon^j \in \{\mu \oplus \lambda_{i_1} \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p} \cdot \varepsilon^{i_p}|\lambda \in \mathbf{B}^n\} \setminus \{\Phi^{\lambda}(\mu)|\lambda \in \mathbf{B}^n\}$, contradiction with (4) again.

(3) is proved. $\square$

Remark 12. *For any function Φ and any $\mu \in \mathbf{B}^n$, Theorem 11 shows that the set μ^{+} is a linear variety (an affine space).*

Definition 13. *A point $\mu \in \mathbf{B}^n$ is called:*

a) ***isolated fixed point****, if $\mu^{-} = \{\mu\}, \mu^{+} = \{\mu\}$;*

b) ***source****, if $\mu^{-} = \{\mu\}, \mu^{+} \neq \{\mu\}$;*

$$(0,\underline{0}) \longrightarrow (\underline{0},1) \longrightarrow (1,1) \qquad\qquad (1,0)$$

Figure 5. Example of predecessors and successors.

*c) **sink**, if* $\mu^- \neq \{\mu\}, \mu^+ = \{\mu\}$;
*d) **transient point**, if* $\mu^- \neq \{\mu\}, \mu^+ \neq \{\mu\}$.

Remark 14. *For a function* Φ, *any point* μ *is in one of the previous situations a),...,d).*

Example 15. *In Figure 5,* $(0,0)$ *is source:* $(0,0)^- = \{(0,0)\}, (0,0)^+ = \{(0,0),(0,1)\}$; $(0,1)$ *is transient:* $(0,1)^- = \{(0,0),(0,1)\}, (0,1)^+ = \{(0,1),(1,1)\}$; $(1,1)$ *is sink:* $(1,1)^- = \{(0,1),(1,1)\}, (1,1)^+ = \{(1,1)\}$; *and* $(1,0)$ *is an isolated fixed point:* $(1,0)^- = (1,0)^+ = \{(1,0)\}$.

Remark 16. *Previously, the function* Φ *was unique and it was kept in mind while using the notations* μ^-, μ^+. *When several functions* $\Phi, \Psi, ...$ *will occur and we shall need to specify to which function we refer, the notations will be* $\mu^-_\Phi, \mu^+_\Psi, ...$

4. Definition of Time-Reversal Symmetry

Theorem 17. *Let* $\Phi, \Psi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$. *The conjunction of the statements*

$$\mu^-_\Phi = \mu^+_\Psi, \tag{5}$$

$$\mu^-_\Psi = \mu^+_\Phi \tag{6}$$

where $\mu \in \mathbf{B}^n$ *is equivalent to the conjunction of the statements*

$$\forall \nu \in \mathbf{B}^n, \exists \lambda \in \mathbf{B}^n, (\Phi^\lambda \circ \Psi^\nu)(\mu) = \mu, \tag{7}$$

$$\forall \lambda \in \mathbf{B}^n, \exists \nu \in \mathbf{B}^n, (\Psi^\nu \circ \Phi^\lambda)(\mu) = \mu \tag{8}$$

where $\mu \in \mathbf{B}^n$ *and it is also equivalent to the conjunction of the statements*

$$\forall \alpha \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_{\_}, \exists \beta \in \widehat{\Pi}_n, \widehat{\Phi}^\beta(\widehat{\Psi}^\alpha(\mu,k),k) = \mu, \tag{9}$$

$$\forall \beta \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_{\_}, \exists \alpha \in \widehat{\Pi}_n, \widehat{\Psi}^\alpha(\widehat{\Phi}^\beta(\mu,k),k) = \mu \tag{10}$$

where $\mu \in \mathbf{B}^n$.

Proof. In order to prove that $((5)$ and $(6))$ is equivalent to $((7)$ and $(8))$ for any $\mu \in \mathbf{B}^n$, we fix an arbitrary such μ. Here is the proof of this equivalence.

(5)$\Longrightarrow$(7). Let $\nu \in \mathbf{B}^n$ and $\Psi^\nu(\mu) = \mu' \in \mu^+_\Psi$. As $\mu' \in \mu^-_\Phi$, $\lambda \in \mathbf{B}^n$ exists with $\Phi^\lambda(\mu') = \mu$, thus (7) holds.

(6)$\Longrightarrow$(8). For $\lambda \in \mathbf{B}^n$ we have $\Phi^\lambda(\mu) = \mu' \in \mu^+_\Phi$. As $\mu' \in \mu^-_\Psi$, some $\nu \in \mathbf{B}^n$ exists with $\Psi^\nu(\mu') = \mu$, showing the truth of (8).

(8)$\Longrightarrow \mu_\Phi^- \subset \mu_\Psi^+$. Let an arbitrary $\mu' \in \mu_\Phi^-$; then $\lambda \in \mathbf{B}^n$ exists such that $\Phi^\lambda(\mu') = \mu$. From (8) we have the existence of $\nu \in \mathbf{B}^n$ with $(\Psi^\nu \circ \Phi^\lambda)(\mu') = \mu' = \Psi^\nu(\mu)$, thus $\mu' \in \mu_\Psi^+$.

(7)$\Longrightarrow \mu_\Psi^+ \subset \mu_\Phi^-$. We take an arbitrary $\mu' \in \mu_\Psi^+$, for which $\nu \in \mathbf{B}^n$ exists with $\mu' = \Psi^\nu(\mu)$. The relation (7) shows the existence of $\lambda \in \mathbf{B}^n$ for which $(\Phi^\lambda \circ \Psi^\nu)(\mu) = \mu = \Phi^\lambda(\mu')$, meaning that $\mu' \in \mu_\Phi^-$.

(7)$\Longrightarrow \mu_\Psi^- \subset \mu_\Phi^+$. For an arbitrary $\mu' \in \mu_\Psi^-$, some $\nu \in \mathbf{B}^n$ exists with $\Psi^\nu(\mu') = \mu$. The relation (7) shows the existence of $\lambda \in \mathbf{B}^n$ with $(\Phi^\lambda \circ \Psi^\nu)(\mu') = \mu' = \Phi^\lambda(\mu)$, thus $\mu' \in \mu_\Phi^+$.

(8)$\Longrightarrow \mu_\Phi^+ \subset \mu_\Psi^-$. Let $\mu' \in \mu_\Phi^+$ arbitrary, thus $\lambda \in \mathbf{B}^n$ exists with $\mu' = \Phi^\lambda(\mu)$. From (8) we have the existence of $\nu \in \mathbf{B}^n$ such that $(\Psi^\nu \circ \Phi^\lambda)(\mu) = \mu = \Psi^\nu(\mu')$, giving $\mu' \in \mu_\Psi^-$.

We prove now that $\forall \mu \in \mathbf{B}^n$,(7)$\Longrightarrow \forall \mu \in \mathbf{B}^n$,(9) and we rewrite for this the hypothesis with the bounded variables slightly modified: $\forall \mu' \in \mathbf{B}^n$,

$$\forall \nu \in \mathbf{B}^n, \exists \lambda \in \mathbf{B}^n, (\Phi^\lambda \circ \Psi^\nu)(\mu') = \mu'. \tag{11}$$

Let in (9) $\mu \in \mathbf{B}^n, \alpha \in \widehat{\Pi}_n$ and $k \in \mathbf{N}_{\_}$ arbitrary and fixed. We have the following possibilities.

Case $k = -1$

We see that

$$\widehat{\Phi}^\beta(\widehat{\Psi}^\alpha(\mu, -1), -1) = \widehat{\Psi}^\alpha(\mu, -1) = \mu$$

is true for $\beta \in \widehat{\Pi}_n$ arbitrary.

Case $k = 0$

(11) written for $\nu = \alpha^0, \lambda = \beta^0, \mu' = \mu$ gives

$$(\Phi^{\beta^0} \circ \Psi^{\alpha^0})(\mu) = \mu, \tag{12}$$

$$\widehat{\Phi}^\beta(\widehat{\Psi}^\alpha(\mu, 0), 0) = (\Phi^{\beta^0} \circ \Psi^{\alpha^0})(\mu) \overset{(12)}{=} \mu.$$

In this case $\beta^1, \beta^2, ... \in \mathbf{B}^n$ are arbitrary such that $\beta \in \widehat{\Pi}_n$.

Case $k = 1$

(11) written for $\nu = \alpha^1, \lambda = \beta^0, \mu' = \Psi^{\alpha^0}(\mu)$ gives

$$(\Phi^{\beta^0} \circ \Psi^{\alpha^1})(\Psi^{\alpha^0}(\mu)) = \Psi^{\alpha^0}(\mu) \tag{13}$$

and (11) written for $\nu = \alpha^0, \lambda = \beta^1, \mu' = \mu$ gives

$$(\Phi^{\beta^1} \circ \Psi^{\alpha^0})(\mu) = \mu. \tag{14}$$

We conclude that

$$\widehat{\Phi}^\beta(\widehat{\Psi}^\alpha(\mu, 1), 1) = (\Phi^{\beta^1} \circ \Phi^{\beta^0} \circ \Psi^{\alpha^1} \circ \Psi^{\alpha^0})(\mu)$$

$$= (\Phi^{\beta^1} \circ (\Phi^{\beta^0} \circ \Psi^{\alpha^1}))(\Psi^{\alpha^0}(\mu)) = \Phi^{\beta^1}((\Phi^{\beta^0} \circ \Psi^{\alpha^1})(\Psi^{\alpha^0}(\mu)))$$

$$\overset{(13)}{=} \Phi^{\beta^1}(\Psi^{\alpha^0}(\mu)) = (\Phi^{\beta^1} \circ \Psi^{\alpha^0})(\mu) \overset{(14)}{=} \mu.$$

In this case we can take $\beta^2, \beta^3, ... \in \mathbf{B}^n$ arbitrarily such that $\beta \in \widehat{\Pi}_n$.

...

Case k

(11) written for $\nu = \alpha^k, \lambda = \beta^0, \mu' = \Psi^{\alpha^{k-1}} \circ ... \circ \Psi^{\alpha^0}(\mu)$ gives

$$(\Phi^{\beta^0} \circ \Psi^{\alpha^k})(\Psi^{\alpha^{k-1}} \circ ... \circ \Psi^{\alpha^0}(\mu)) = \Psi^{\alpha^{k-1}} \circ ... \circ \Psi^{\alpha^0}(\mu), \tag{15}$$

(11) written for $\nu = \alpha^{k-1}, \lambda = \beta^1, \mu' = \Psi^{\alpha^{k-2}} \circ ... \circ \Psi^{\alpha^0}(\mu)$ gives

$$(\Phi^{\beta^1} \circ \Psi^{\alpha^{k-1}})(\Psi^{\alpha^{k-2}} \circ ... \circ \Psi^{\alpha^0}(\mu)) = \Psi^{\alpha^{k-2}} \circ ... \circ \Psi^{\alpha^0}(\mu), \tag{16}$$

...

(11) written for $\nu = \alpha^0, \lambda = \beta^k, \mu' = \mu$ gives

$$(\Phi^{\beta^k} \circ \Psi^{\alpha^0})(\mu) = \mu. \tag{17}$$

We infer:

$$\widehat{\Phi}^{\beta}(\widehat{\Psi}^{\alpha}(\mu, k), k) = (\Phi^{\beta^k} \circ \Phi^{\beta^{k-1}} \circ ... \circ \Phi^{\beta^0} \circ \Psi^{\alpha^k} \circ \Psi^{\alpha^{k-1}} \circ ... \circ \Psi^{\alpha^0})(\mu)$$

$$= (\Phi^{\beta^k} \circ \Phi^{\beta^{k-1}} \circ ... \circ \Phi^{\beta^1} \circ (\Phi^{\beta^0} \circ \Psi^{\alpha^k}))(\Psi^{\alpha^{k-1}} \circ ... \circ \Psi^{\alpha^0}(\mu))$$

$$\overset{(15)}{=} (\Phi^{\beta^k} \circ \Phi^{\beta^{k-1}} \circ ... \circ \Phi^{\beta^1})(\Psi^{\alpha^{k-1}} \circ ... \circ \Psi^{\alpha^0}(\mu))$$

$$= (\Phi^{\beta^k} \circ \Phi^{\beta^{k-1}} \circ ... \circ \Phi^{\beta^2} \circ (\Phi^{\beta^1} \circ \Psi^{\alpha^{k-1}}))(\Psi^{\alpha^{k-2}} \circ ... \circ \Psi^{\alpha^0}(\mu))$$

$$\overset{(16)}{=} (\Phi^{\beta^k} \circ \Phi^{\beta^{k-1}} \circ ... \circ \Phi^{\beta^2})(\Psi^{\alpha^{k-2}} \circ ... \circ \Psi^{\alpha^0}(\mu)) = ...$$

$$... = (\Phi^{\beta^k} \circ \Psi^{\alpha^0})(\mu) \overset{(17)}{=} \mu.$$

In this case we can take $\beta^{k+1}, \beta^{k+2}, ... \in \mathbf{B}^n$ arbitrarily such that $\beta \in \widehat{\Pi}_n$.

$\forall \mu \in \mathbf{B}^n$,(8)$\Longrightarrow$ $\forall \mu \in \mathbf{B}^n$,(10) is similar with the proof of $\forall \mu \in \mathbf{B}^n$,(7)$\Longrightarrow$ $\forall \mu \in \mathbf{B}^n$,(9).

For $\mu \in \mathbf{B}^n$ arbitrary, (9)$\Longrightarrow$(7) is obvious if we take in (9) $k = 0, \alpha^0 = \nu, \beta^0 = \lambda$ and the implication (10)$\Longrightarrow$(8) with $\mu \in \mathbf{B}^n$ arbitrary also takes place similarly. $\square$

Definition 18. *If one of the properties*

$$\forall \mu \in \mathbf{B}^n, (5) \text{ and } (6),$$

$$\forall \mu \in \mathbf{B}^n, (7) \text{ and } (8),$$

$$\forall \mu \in \mathbf{B}^n, (9) \text{ and } (10)$$

is fulfilled, we say that the ***time-reversal symmetry*** *of the functions* Φ *and* Ψ *holds.*

Example 19. *We notice the time-reversal symmetry of* $1_{\mathbf{B}^n}$ *and* $1_{\mathbf{B}^n}$*, see Figure 6, where* $1_{\mathbf{B}^n} : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *is the identity function. All* $\mu \in \mathbf{B}^n$ *fulfill* $\mu^+ = \mu^- = \{\mu\}$*, i.e. they are isolated fixed points.*

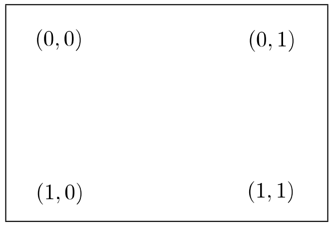

Figure 6. The time-reversal symmetry of $1_{\mathbf{B}^n}$ and $1_{\mathbf{B}^n}$.

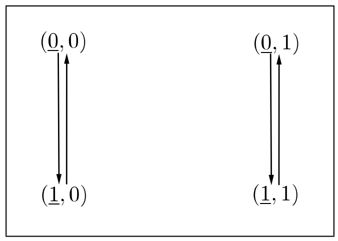

Figure 7. The time-reversal symmetry of Φ and Φ.

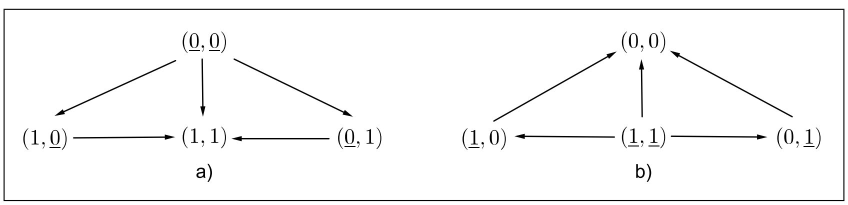

Figure 8. The time-reversal symmetry of two constant functions.

Example 20. *Let the function Φ from Figure 7. We notice the time-reversal symmetry of Φ with Φ, and also the fact that all $\mu \in \mathbf{B}^2$ are transient in this example:* $(0,0)^+ = (0,0)^- = (1,0)^+ = (1,0)^- = \{(0,0),(1,0)\}$, $(0,1)^+ = (0,1)^- = (1,1)^+ = (1,1)^- = \{(0,1), (1,1)\}$.

Example 21. *The functions $\Phi, \Psi : \mathbf{B}^2 \longrightarrow \mathbf{B}^2$ from Figure 8 a), b) are constant:* $\forall \mu \in \mathbf{B}^2$,

$$\Phi(\mu) = (1,1),$$

$$\Psi(\mu) = (0,0).$$

Notice the existence of the same arrows at a), b), with different senses however, that show the meaning of time-reversal symmetry. The situation was the same in the previous two cases, but less obvious, due to the fact that the symmetrical functions coincided.

Example 22. *We see in Figure 9 a), b) the time-reversal symmetry of the functions Φ, Ψ; these functions are neither constant, nor equal.*

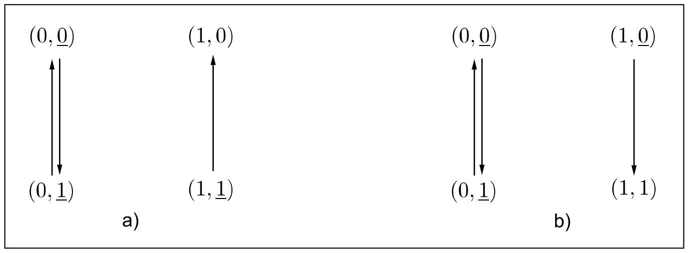

Figure 9. The time-reversal symmetry of two functions that are neither constant, nor equal.

5. Properties

Remark 23. *The time-reversal symmetry of* $\Phi, \Psi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *does not imply the satisfaction of*

$$\forall \lambda \in \mathbf{B}^n, \exists \nu \in \mathbf{B}^n, (\Phi^\lambda \circ \Psi^\nu)(\mu) = \mu \tag{18}$$

or

$$\forall \nu \in \mathbf{B}^n, \exists \lambda \in \mathbf{B}^n, (\Psi^\nu \circ \Phi^\lambda)(\mu) = \mu, \tag{19}$$

to be compared with (7) and (8). For this, we notice that in Figure 8 with Φ *at a) and* Ψ *at b) we have* $(0,1)^+_\Psi = \{(0,0),(0,1)\}$ *and for* $\lambda = (1,1)$ *we get*

$$\Phi^\lambda((0,1)^+_\Psi) = \Phi(\{(0,0),(0,1)\}) = \{(1,1)\} \neq \{(0,1)\},$$

in other words (18) is false. The Theorem that follows shows that these properties take place under a different form.

Theorem 24. *For* $\Phi, \Psi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *and* $\mu \in \mathbf{B}^n$,

$$\mu^-_\Phi = \mu^+_\Psi \tag{20}$$

implies

$$\forall \lambda \in \mathbf{B}^n, (\Phi^\lambda)^{-1}(\mu) \neq \varnothing \Longrightarrow \exists \nu \in \mathbf{B}^n, (\Phi^\lambda \circ \Psi^\nu)(\mu) = \mu, \tag{21}$$

and

$$\mu^-_\Psi = \mu^+_\Phi \tag{22}$$

implies

$$\forall \nu \in \mathbf{B}^n, (\Psi^\nu)^{-1}(\mu) \neq \varnothing \Longrightarrow \exists \lambda \in \mathbf{B}^n, (\Psi^\nu \circ \Phi^\lambda)(\mu) = \mu. \tag{23}$$

Proof. (20)$\Longrightarrow$(21). For $\lambda \in \mathbf{B}^n$ arbitrary, we suppose that $(\Phi^\lambda)^{-1}(\mu) \neq \varnothing$ and let $\mu' \in (\Phi^\lambda)^{-1}(\mu)$ be arbitrary too. As $\mu' \in \mu^-_\Phi = \mu^+_\Psi$, some $\nu \in \mathbf{B}^n$ exists with $\mu' = \Psi^\nu(\mu)$. We have $(\Phi^\lambda \circ \Psi^\nu)(\mu) = \Phi^\lambda(\mu') = \mu$, thus (21) is true.

(22)$\Longrightarrow$(23). Similar. $\square$

Example 25. *We continue the example from Remark 23. Indeed, for* $\lambda = (1,1)$ *we have in Figure 8 that* $\Phi^{-1}(0,1) = \varnothing$. *We can take however* $\lambda = (0,1)$ *giving* $(\Phi^{(0,1)})^{-1}(0,1) = \{(0,0),(0,1)\}$*; then for* $\nu = (1,1)$ *and* $\nu = (1,0)$ *we have*

$$(\Phi^{(0,1)} \circ \Psi^{(1,1)})(0,1) = \Phi^{(0,1)}(0,0) = (0,1) = \Phi^{(0,1)}(0,1) = (\Phi^{(0,1)} \circ \Psi^{(1,0)})(0,1).$$

Theorem 26. *Let the functions* $\Phi, \Psi, \Gamma : \mathbf{B}^n \longrightarrow \mathbf{B}^n$. *The time-reversal symmetry of* Φ *and* Ψ, *together with the time-reversal symmetry of* Φ *and* Γ *imply* $\Psi = \Gamma$.

Proof. We suppose against all reason the contrary, that $\Psi \neq \Gamma$, meaning the existence of $\mu \in \mathbf{B}^n$ with $\Psi(\mu) \neq \Gamma(\mu)$. We infer the existence of the sets $I, J \subset \{1, ..., n\}$ with the property that $\Psi(\mu) = \mu \oplus \bigoplus_{i \in I} \varepsilon^i$, $\Gamma(\mu) = \mu \oplus \bigoplus_{i \in J} \varepsilon^i$ and $I \neq J$. Without loss of generality, we can suppose the existence of some $i \in I \setminus J$. We infer that $\mu \oplus \varepsilon^i \in \mu^+_\Psi$, $\mu \oplus \varepsilon^i \notin \mu^+_\Gamma$, and this represents a contradiction with the hypothesis stating that $\mu^+_\Psi = \mu^-_\Phi = \mu^+_\Gamma$. □

6. The Technical Condition of Proper Operation

Remark 27. *In the study of the asynchronous circuits, the situation when multiple coordinates of the flow can change at the same time is called a race. The coordinates* $\mu_1, ..., \mu_n$ *are 'racing' to see which one can change first. In Figure 1 we have presented the case of a flow that has a race in* $(0,0)$ *and we have seen that the outcome of the race affected critically the work of the circuit (including its stability). To avoid the races that could occur,* Φ *is sometimes specified so that only one coordinate of* μ *can change; such a circuit is called race-free and we also say that* Φ *fulfills the technical condition of proper operation.*

Theorem 28. *The following statements are equivalent:*

a) $\forall \mu \in \mathbf{B}^n$, *one of the following properties is true:*

$$\Phi(\mu) = \mu, \tag{24}$$

$$\exists i \in \{1, ..., n\}, \Phi(\mu) = \mu \oplus \varepsilon^i. \tag{25}$$

b) $\forall \mu \in \mathbf{B}^n$, *one of the following properties is true:*

$$\Phi^{-1}(\mu) = \varnothing, \tag{26}$$

$$\Phi^{-1}(\mu) = \{\mu\}, \tag{27}$$

$$\exists i \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu \oplus \varepsilon^i\}, \tag{28}$$

$$\exists i \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu, \mu \oplus \varepsilon^i\}, \tag{29}$$

$$\exists i \in \{1, ..., n\}, \exists j \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu \oplus \varepsilon^i, \mu \oplus \varepsilon^j\}, \tag{30}$$

$$\exists i \in \{1, ..., n\}, \exists j \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu, \mu \oplus \varepsilon^i, \mu \oplus \varepsilon^j\}, \tag{31}$$

$$...$$

$$\exists i_1 \in \{1, ..., n\}, ..., \exists i_n \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu \oplus \varepsilon^{i_1}, ..., \mu \oplus \varepsilon^{i_n}\}, \tag{32}$$

$$\exists i_1 \in \{1, ..., n\}, ..., \exists i_n \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu, \mu \oplus \varepsilon^{i_1}, ..., \mu \oplus \varepsilon^{i_n}\} \tag{33}$$

Proof. a)$\Longrightarrow$b) Let us fix an arbitrary $\mu \in \mathbf{B}^n$ and we suppose against all reason that (26),...,(33) are all false. This means the existence of $p \in \{2, ..., n\}$ and $i_1, ..., i_p \in \{1, ..., n\}$ distinct such that $\mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p} \in \Phi^{-1}(\mu)$. Then $\Phi(\mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}) = \mu$ and (24), (25) are both false for $\mu' = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}$, contradiction.

b)$\Longrightarrow$a) We suppose against all reason that a) is false. This means the existence of $\mu \in \mathbf{B}^n, p \in \{2, ..., n\}$ and $i_1, ..., i_p \in \{1, ..., n\}$ distinct such that $\Phi(\mu) = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}$. We infer that $\mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p} \in \Phi^{-1}(\mu)$, i.e. (26),...,(33) are all false, contradiction. □

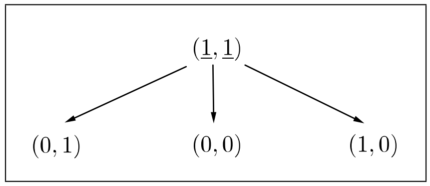

Figure 10. Function that does not fulfill the technical condition of proper operation.

Definition 29. *The function Φ is said to fulfill the* ***technical condition of proper operation*** *if one of the previous properties a), b) holds.*

Remark 30. *The technical condition of proper operation states that for all μ, the $n-$tuples μ and $\Phi(\mu)$ differ on at most one coordinate.*

Example 31. *The identity $1_{\mathbf{B}^n} : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ fulfills the technical condition of proper operation, since all $\mu \in \mathbf{B}^n$ are fixed points of $1_{\mathbf{B}^n}$.*

Example 32. *The functions from Figure 5, Figure 7 and Figure 9 a), b) fulfill the technical condition of proper operation too.*

Example 33. *The function Φ from Figure 10 does not fulfill the technical condition of proper operation, since $\Phi(1,1) = (0,0)$.*

7. The Dynamics under the Technical Condition of Proper Operation

Theorem 34. *[4] If Φ fulfills the technical condition of proper operation and $\widehat{\Phi}^{\alpha}(\mu, k_1) = \mu'$, then one of the following possibilities is true:*

a) $\Phi(\mu') = \mu'$ and $\forall k \geq k_1$,

$$\widehat{\Phi}^{\alpha}(\mu, k) = \mu' = \Phi(\mu');$$

b) $i \in \{1, ..., n\}$ exists such that $\Phi(\mu') = \mu' \oplus \varepsilon^i$ and either

$$\widehat{\Phi}^{\alpha}(\mu, k_1 + 1) = \mu' \oplus \varepsilon^i = \Phi(\mu'),$$

or $k_2 \geq k_1 + 2$ exists with

$$\widehat{\Phi}^{\alpha}(\mu, k_1 + 1) = ... = \widehat{\Phi}^{\alpha}(\mu, k_2 - 1) = \mu',$$

$$\widehat{\Phi}^{\alpha}(\mu, k_2) = \mu' \oplus \varepsilon^i = \Phi(\mu').$$

Notation 35. *We define $\Phi^{(k)} : \mathbf{B}^n \longrightarrow \mathbf{B}^n$, $k \in \mathbf{N}$ by $\forall \mu \in \mathbf{B}^n$,*

$$\Phi^{(k)}(\mu) = \begin{cases} \mu, if\ k = 0, \\ \Phi(\Phi^{(k-1)}(\mu)), if\ k \geq 1. \end{cases}$$

Theorem 36. *If Φ fulfills the technical condition of proper operation, then $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \widehat{\Pi}_n$, a sequence $-1 = j_0 < j_1 < ... < j_k < ...$ exists having the property that $\forall k \in \mathbf{N}$,*

a) $\widehat{\Phi}^{\alpha}(\mu, j_k) = \Phi^{(k)}(\mu)$;

b) if $j_{k+1} - j_k \geq 2$, then $\forall j \in \{j_k, j_k + 1, ..., j_{k+1} - 1\}, \widehat{\Phi}^{\alpha}(\mu, j) = \Phi^{(k)}(\mu)$.

Proof. We use the induction on k. Let $k = 0$, for which we have $j_0 = -1$ and the statement a) is true under the form $\widehat{\Phi}^{\alpha}(\mu, -1) = \mu = \Phi^{(0)}(\mu)$. In order to prove b), we use the hypothesis that Φ satisfies the technical condition of proper operation and we have two possibilities.

Case $\Phi(\mu) = \mu$

As $\forall j \geq j_0, \widehat{\Phi}^{\alpha}(\mu, j) = \mu = \Phi(\mu)$, from Theorem 34 a), we choose $j_1 > j_0$ arbitrarily and the statement b) takes place trivially: if $j_1 - j_0 \geq 2$, then $\forall j \in \{j_0, j_0 + 1, ..., j_1 - 1\}, \widehat{\Phi}^{\alpha}(\mu, j) = \mu = \Phi^{(0)}(\mu)$.

Case $\Phi(\mu) = \mu \oplus \varepsilon^i, i \in \{1, ..., n\}$

If $\widehat{\Phi}^{\alpha}(\mu, 0) = \mu \oplus \varepsilon^i = \Phi(\mu)$, from Theorem 34 b), we take $j_1 = 0$ and the statement b) is trivially true. Otherwise, from Theorem 34 b), we take $j_1 \geq j_0 + 2$ such that

$$\widehat{\Phi}^{\alpha}(\mu, 0) = ... = \widehat{\Phi}^{\alpha}(\mu, j_1 - 1) = \mu, \tag{34}$$
$$\widehat{\Phi}^{\alpha}(\mu, j_1) = \mu \oplus \varepsilon^i = \Phi(\mu).$$

The statement b) results from (34).

The proof by induction continues, we suppose that the statements of the Theorem are true for k and we prove them for $k + 1$. $\square$

Remark 37. *Theorem 34 gives the meaning of the technical condition of proper operation. In the situation when we do not know the time instants and the order in which the coordinate functions $\Phi_1, ..., \Phi_n$ are computed, what we surely know is that if $\widehat{\Phi}^{\alpha}(\mu, k_1) = \mu'$, then, independently on the values $\alpha^k \in \mathbf{B}^n, k \geq k_1 + 1$, some $k_2 \geq k_1 + 1$ exists such that $\widehat{\Phi}^{\alpha}(\mu, k_2) = \Phi(\mu')$.*

Remark 38. *Theorem 36 shows that if the technical condition of proper operation is fulfilled, then $\widehat{\Phi}^{\alpha}(\mu, \cdot)$ behaves like a dynamical system.*

8. Time-Reversal Symmetry vs. the Technical Condition of Proper Operation

Theorem 39. *If Φ fulfills the technical condition of proper operation and the time-reversal symmetry of Φ, Ψ holds, then Ψ fulfills the technical condition of proper operation.*

Proof. We suppose against all reason that Ψ does not satisfy the technical condition of proper operation. Some μ, $p \geq 2$ and $i_1, ..., i_p \in \{1, ..., n\}$ distinct exist then such that $\Psi(\mu) = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}$. Since $\mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p} \in \mu_{\Psi}^{+} = \mu_{\Phi}^{-}$, some $\lambda \in \mathbf{B}^n$ exists such that $\Phi^{\lambda}(\mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}) = \mu$ and, with the notation $\mu' = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}$, we get $\Phi^{\lambda}(\mu') = \mu' \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}$. We have obtained the existence of the set I such that $\{i_1, ..., i_p\} \subset I \subset \{1, ..., n\}$ and $\Phi(\mu') = \mu' \oplus \bigoplus_{i \in I} \varepsilon^i$, where $card(I) \geq 2$. This last assertion represents a contradiction with the request that Φ fulfills the technical condition of proper operation. $\square$

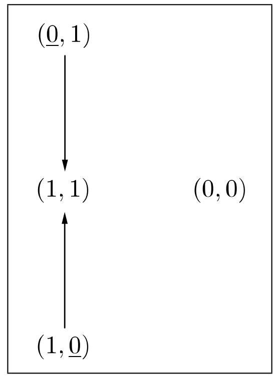

Figure 11. Function Φ that fulfills the technical condition of proper operation; the time-reversal symmetry of Φ and Ψ holds for no function Ψ.

Remark 40. *Functions Φ exist that fulfill the technical condition of proper operation and the time-reversal symmetry of Φ and Ψ holds for no function Ψ, see Figure 11. A consequence of this remark is that, in order to study the time-reversal symmetry under the technical condition of proper operation, stronger requests than the fulfillment of the technical condition of proper operation are necessary.*

Notation 41. *We denote, for $\mu \in \mathbf{B}^n$, with $\Delta_\Phi(\mu)$, the disjunction of the following four properties:*

$$\Phi^{-1}(\mu) = \varnothing, \tag{35}$$

$$\Phi^{-1}(\mu) = \{\mu\}, \tag{36}$$

$$\exists i \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu \oplus \varepsilon^i\}, \tag{37}$$

$$\exists i \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu, \mu \oplus \varepsilon^i\}, \tag{38}$$

coinciding with (26),...,(29).

Remark 42. *If $\forall \mu \in \mathbf{B}^n$, we have $\Delta_\Phi(\mu)$, then Φ fulfills the technical condition of proper operation, see Theorem 28 and Definition 29.*

Theorem 43. *We suppose that for all $\mu \in \mathbf{B}^n$, $\Delta_\Phi(\mu)$ holds. Then $\forall \mu \in \mathbf{B}^n$,*

a) $\Phi^{-1}(\mu) = \varnothing$ implies that μ is a source, $\mu_\Phi^- = \{\mu\}$ and $\exists i \in \{1, ..., n\}, \mu_\Phi^+ = \{\mu, \mu \oplus \varepsilon^i\}$;

b) $\Phi^{-1}(\mu) = \{\mu\}$ implies that μ is an isolated fixed point, $\mu_\Phi^- = \{\mu\}$ and $\mu_\Phi^+ = \{\mu\}$;

c) $\exists i \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu \oplus \varepsilon^i\}$ implies that μ is a transient point, $\mu_\Phi^- = \{\mu, \mu \oplus \varepsilon^i\}$ and $\exists j \in \{1, ..., n\}, \mu_\Phi^+ = \{\mu, \mu \oplus \varepsilon^j\}$;

d) $\exists i \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu, \mu \oplus \varepsilon^i\}$ implies that μ is a sink, $\mu_\Phi^- = \{\mu, \mu \oplus \varepsilon^i\}$ and $\mu_\Phi^+ = \{\mu\}$.

Proof. We fix an arbitrary $\mu \in \mathbf{B}^n$.

a) We suppose against all reason that $\mu_\Phi^- \neq \{\mu\}$, i.e. some $\mu' \neq \mu$ and $\lambda \in \mathbf{B}^n$ exist such that $\Phi^\lambda(\mu') = \mu$. As Φ fulfills the technical condition of proper operation, we have the following cases.

Case $\Phi(\mu') = \mu'$

In this case $\forall\lambda \in \mathbf{B}^n$, we have $\Phi^\lambda(\mu') = \mu' \neq \mu$, contradiction with the supposition that $\mu' \in \mu_\Phi^-$.

Case $\exists i \in \{1, ..., n\}, \Phi(\mu') = \mu' \oplus \varepsilon^i$

For an arbitrary $\lambda \in \mathbf{B}^n$, we get the next possibilities. If $\lambda_i = 0$, then $\Phi^\lambda(\mu') = \mu' \neq \mu$, contradiction. And if $\lambda_i = 1$, then $\Phi(\mu') = \Phi^\lambda(\mu') = \mu' \oplus \varepsilon^i \neq \mu$, contradiction with the supposition that $\mu' \in \mu_\Phi^-$ (if we would have $\mu' \oplus \varepsilon^i = \mu$, then $\mu' \in \Phi^{-1}(\mu)$, representing another contradiction, with the hypothesis $\Phi^{-1}(\mu) = \varnothing$).

We have proved that $\mu_\Phi^- = \{\mu\}$.

In order to show the second statement, we know that two possibilities exist, from the fact that Φ fulfills the technical condition of proper operation: either $\Phi(\mu) = \mu$, which is not the case, since $\mu \in \Phi^{-1}(\mu)$ and then we get a contradiction with the supposition that $\Phi^{-1}(\mu) = \varnothing$, or $\exists i \in \{1, ..., n\}, \Phi(\mu) = \mu \oplus \varepsilon^i$, which proves that $\mu_\Phi^+ = \{\mu, \mu \oplus \varepsilon^i\}$.

b) Like at a), let us suppose against all reason that $\mu_\Phi^- \neq \{\mu\}$, meaning that $\mu' \neq \mu$ and $\lambda \in \mathbf{B}^n$ exist such that $\Phi^\lambda(\mu') = \mu$. Since Φ fulfills the technical condition of proper operation, we get the following possibilities.

Case $\Phi(\mu') = \mu'$

For any $\lambda \in \mathbf{B}^n$, $\Phi^\lambda(\mu') = \mu' \neq \mu$ and this contradicts the supposition that $\mu' \in \mu_\Phi^-$.

Case $\exists i \in \{1, ..., n\}, \Phi(\mu') = \mu' \oplus \varepsilon^i$

Let $\lambda \in \mathbf{B}^n$ arbitrary. If $\lambda_i = 0$, then $\Phi^\lambda(\mu') = \mu' \neq \mu$, contradiction with the request $\mu' \in \mu_\Phi^-$. If $\lambda_i = 1$, when $\Phi(\mu') = \Phi^\lambda(\mu') = \mu' \oplus \varepsilon^i \neq \mu$, we obtain a contradiction with the supposition that $\mu' \in \mu_\Phi^-$ (if $\mu' \oplus \varepsilon^i = \mu$, then $\mu' \in \Phi^{-1}(\mu)$, representing a contradiction with the hypothesis $\Phi^{-1}(\mu) = \{\mu\}$).

The conclusion is that $\mu_\Phi^- = \{\mu\}$.

The fact that $\mu_\Phi^+ = \{\Phi^\lambda(\mu) | \lambda \in \mathbf{B}^n\} = \{\mu\}$ is obvious, as far as $\Phi(\mu) = \mu$ implies $\forall\lambda \in \mathbf{B}^n, \Phi^\lambda(\mu) = \mu$.

c) It is obvious that $\mu, \mu \oplus \varepsilon^i \in \mu_\Phi^-$. We suppose against all reason that $\mu' \in \mu_\Phi^-$ exists, $\mu' \notin \{\mu, \mu \oplus \varepsilon^i\}$, implying the existence of $\lambda \in \mathbf{B}^n$ with $\Phi^\lambda(\mu') = \mu$. The following possibilities are a consequence of the fact that Φ fulfills the technical condition of proper operation.

Case $\Phi(\mu') = \mu'$

Then $\forall\lambda \in \mathbf{B}^n, \Phi^\lambda(\mu') = \Phi(\mu') = \mu' \neq \mu$, contradiction with the supposition $\mu' \in \mu_\Phi^-$.

Case $\exists j \in \{1, ..., n\}, \Phi(\mu') = \mu' \oplus \varepsilon^j$

If $\lambda_j = 0$, then $\Phi^\lambda(\mu') = \mu' \neq \mu$, contradiction with the supposition $\mu' \in \mu_\Phi^-$.

If $\lambda_j = 1$, then $\Phi(\mu') = \Phi^\lambda(\mu') = \mu' \oplus \varepsilon^j$. The supposition that $\mu' \oplus \varepsilon^j = \mu$ implies $\mu' \in \Phi^{-1}(\mu) = \{\mu \oplus \varepsilon^i\}$, contradiction with the request $\mu' \notin \{\mu, \mu \oplus \varepsilon^i\}$. And if $\mu' \oplus \varepsilon^j \neq \mu$, then a contradiction with the request $\mu' \in \mu_\Phi^-$ follows.

We have obtained that $\mu_\Phi^- = \{\mu, \mu \oplus \varepsilon^i\}$.

The satisfaction of the technical condition of proper operation gives the next cases.

Case $\Phi(\mu) = \mu$

This is impossible, since it implies $\mu \in \Phi^{-1}(\mu) = \{\mu \oplus \varepsilon^i\}$.

Case $\exists j \in \{1, ..., n\}, \Phi(\mu) = \mu \oplus \varepsilon^j$

We infer that $\mu_\Phi^+ = \{\mu, \mu \oplus \varepsilon^j\}$ holds indeed.

d) Like at c), the fact that $\mu, \mu\oplus\varepsilon^i \in \mu_\Phi^-$ is obvious. Let us suppose against all reason that $\mu' \in \mu_\Phi^-$ exists, $\mu' \notin \{\mu, \mu\oplus\varepsilon^i\}$, meaning the existence of $\lambda \in \mathbf{B}^n$ with $\Phi^\lambda(\mu') = \mu$. But Φ fulfills the technical condition of proper operation, wherefrom we obtain the following cases.

Case $\Phi(\mu') = \mu'$

We infer $\forall\lambda \in \mathbf{B}^n, \Phi^\lambda(\mu') = \Phi(\mu') = \mu' \neq \mu$, contradiction with our demand $\mu' \in \mu_\Phi^-$.

Case $\exists j \in \{1, ..., n\}, \Phi(\mu') = \mu' \oplus \varepsilon^j$

If $\lambda_j = 0$, then $\Phi^\lambda(\mu') = \mu' \neq \mu$, giving a contradiction with the request $\mu' \in \mu_\Phi^-$.

If $\lambda_j = 1$, then $\Phi(\mu') = \Phi^\lambda(\mu') = \mu' \oplus \varepsilon^j$. In case that $\mu' \oplus \varepsilon^j = \mu$ we infer $\mu' \in \Phi^{-1}(\mu) = \{\mu, \mu\oplus\varepsilon^i\}$, contradiction with the hypothesis $\mu' \notin \{\mu, \mu\oplus\varepsilon^i\}$. And in case that $\mu' \oplus \varepsilon^j \neq \mu$, a contradiction occurs with the request $\mu' \in \mu_\Phi^-$.

We have proved that $\mu_\Phi^- = \{\mu, \mu\oplus\varepsilon^i\}$.

As $\mu \in \Phi^{-1}(\mu)$, we get $\Phi(\mu) = \mu$ and $\mu_\Phi^+ = \{\mu\}$ holds, like at b). $\square$

Theorem 44. *We suppose that* $\forall\mu \in \mathbf{B}^n$, *we have* $\Delta_\Phi(\mu)$ *and we define* $\Psi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *by* $\forall\mu \in \mathbf{B}^n$,

$$\Psi(\mu) = \begin{cases} \mu, \text{ if } \Phi^{-1}(\mu) = \varnothing, \\ \mu, \text{ if } \Phi^{-1}(\mu) = \{\mu\}, \\ \mu\oplus\varepsilon^i, \text{ if } \exists i \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu\oplus\varepsilon^i\}, \\ \mu\oplus\varepsilon^i, \text{ if } \exists i \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu, \mu\oplus\varepsilon^i\}. \end{cases} \tag{39}$$

Then

a) the time-reversal symmetry of Φ *and* Ψ *holds;*

b) for any $\mu \in \mathbf{B}^n$, $\Delta_\Psi(u)$ *is true.*

Proof. We fix an arbitrary $\mu \in \mathbf{B}^n$.

a) We have four possibilities. When treating them, we use Theorem 43.

Case i) μ is a source for Φ, when $\Psi(\mu) = \mu, \Phi^{-1}(\mu) = \varnothing$,

$$\mu_\Phi^- = \{\mu\} = \mu_\Psi^+,$$

$$\exists i \in \{1, ..., n\}, \mu_\Phi^+ = \{\mu, \mu\oplus\varepsilon^i\}$$

and we prove $\mu_\Psi^- = \{\mu, \mu\oplus\varepsilon^i\}$.

$\{\mu, \mu\oplus\varepsilon^i\} \subset \mu_\Psi^-$. In this case $i \in \{1, ..., n\}$ exists with $\Phi(\mu) = \mu\oplus\varepsilon^i$ and, because $\mu \in \Phi^{-1}(\mu\oplus\varepsilon^i)$, we have that $\Psi(\mu\oplus\varepsilon^i) = \mu$. It has resulted that $\mu, \mu\oplus\varepsilon^i \in \mu_\Psi^-$.

$\mu_\Psi^- \subset \{\mu, \mu\oplus\varepsilon^i\}$. We suppose against all reason that $\mu' \in \mu_\Psi^-$ exists, $\mu' \neq \mu, \mu' \neq \mu\oplus\varepsilon^i$, in other words $\mu'' \in \mathbf{B}^n$ and $\nu \in \mathbf{B}^n$ exist such that $\Psi(\mu') = \mu''$ and $\Psi^\nu(\mu') = \mu$. The situation $\mu' = \mu''$ is impossible, as far as it implies the contradiction $\mu' = \Psi(\mu') = \Psi^\nu(\mu') = \mu$, thus $\mu' \neq \mu''$. From the definition of Ψ, see (39), we get the existence of $j \in \{1, ..., n\}$ with $\mu'' = \mu' \oplus \varepsilon^j$, resulting further that either $\Phi^{-1}(\mu') = \{\mu'\oplus\varepsilon^j\}$ or $\Phi^{-1}(\mu') = \{\mu', \mu'\oplus\varepsilon^j\}$ holds. We have $\{\Psi^\nu(\mu') | \nu \in \mathbf{B}^n\} = \{\mu', \mu'\oplus\varepsilon^j\}$ and $\mu \in \{\mu', \mu'\oplus\varepsilon^j\}$.

But $\mu = \mu'$ is impossible, from the way that we have chosen μ' and the only possibility becomes $\mu = \mu' \oplus \varepsilon^j$.

Case $\Phi^{-1}(\mu') = \{\mu' \oplus \varepsilon^j\}$, i.e. $\Phi^{-1}(\mu \oplus \varepsilon^j) = \{\mu\}$

We infer $\Phi(\mu) = \mu \oplus \varepsilon^j = \mu \oplus \varepsilon^i, j = i$ and $\mu' = \mu \oplus \varepsilon^i$, contradiction.

Case $\Phi^{-1}(\mu') = \{\mu', \mu' \oplus \varepsilon^j\}$, i.e. $\Phi^{-1}(\mu \oplus \varepsilon^j) = \{\mu \oplus \varepsilon^j, \mu\}$

Once again the fact that $\Phi(\mu) = \mu \oplus \varepsilon^j = \mu \oplus \varepsilon^i$ implies $j = i$ and $\mu' = \mu \oplus \varepsilon^i$, representing a contradiction.

It has resulted that such a μ' does not exist.

Case ii) μ is an isolated fixed point of Φ, when $\Psi(\mu) = \mu, \Phi^{-1}(\mu) = \{\mu\}$,

$$\mu_\Phi^- = \{\mu\} = \mu_\Psi^+,$$

$$\mu_\Phi^+ = \{\mu\}$$

and we prove $\mu_\Psi^- = \{\mu\}$.

$\{\mu\} \subset \mu_\Psi^-$. Obvious.

$\mu_\Psi^- \subset \{\mu\}$. We suppose against all reason that the inclusion does not take place and let $\mu' \neq \mu$ with the property that $\mu'' \in \mathbf{B}^n$ and $\nu \in \mathbf{B}^n$ exist with $\Psi(\mu') = \mu''$ and $\Psi^\nu(\mu') = \mu$.

The situation $\mu' = \mu''$ gives the contradiction $\mu' = \Psi(\mu') = \Psi^\nu(\mu') = \mu$, therefore $\mu' \neq \mu''$.

From the way that Ψ was defined in (39) some $j \in \{1, ..., n\}$ exists with $\mu'' = \mu' \oplus \varepsilon^j$ and we have that either $\Phi^{-1}(\mu') = \{\mu' \oplus \varepsilon^j\}$ or $\Phi^{-1}(\mu') = \{\mu', \mu' \oplus \varepsilon^j\}$ is true. We infer $\{\Psi^\nu(\mu') | \nu \in \mathbf{B}^n\} = \{\mu', \mu' \oplus \varepsilon^j\}$ and $\mu \in \{\mu', \mu' \oplus \varepsilon^j\}$, where the only possibility is $\mu = \mu' \oplus \varepsilon^j$.

Case $\Phi^{-1}(\mu') = \{\mu' \oplus \varepsilon^j\}$, i.e. $\Phi^{-1}(\mu \oplus \varepsilon^j) = \{\mu\}$

This implies the contradiction $\Phi(\mu) = \mu \oplus \varepsilon^j$.

Case $\Phi^{-1}(\mu') = \{\mu', \mu' \oplus \varepsilon^j\}$, i.e. $\Phi^{-1}(\mu \oplus \varepsilon^j) = \{\mu \oplus \varepsilon^j, \mu\}$

Once again $\Phi(\mu) = \mu \oplus \varepsilon^j$ represents a contradiction.

The conclusion is that such a μ' does not exist.

Case iii) μ is a transient point of Φ, some $i \in \{1, ..., n\}$ exists such that $\Psi(\mu) = \mu \oplus \varepsilon^i, \Phi^{-1}(\mu) = \{\mu \oplus \varepsilon^i\}$,

$$\mu_\Phi^- = \{\mu, \mu \oplus \varepsilon^i\} = \mu_\Psi^+,$$

$$\exists j \in \{1, ..., n\}, \mu_\Phi^+ = \{\mu, \mu \oplus \varepsilon^j\}$$

and we show that $\mu_\Psi^- = \{\mu, \mu \oplus \varepsilon^j\}$.

$\{\mu, \mu \oplus \varepsilon^j\} \subset \mu_\Psi^-$. Since $j \in \{1, ..., n\}$ exists such that $\Phi(\mu) = \mu \oplus \varepsilon^j$, we get $\Psi(\mu \oplus \varepsilon^j) = \mu$ thus $\mu, \mu \oplus \varepsilon^j \in \mu_\Psi^-$.

$\mu_\Psi^- \subset \{\mu, \mu \oplus \varepsilon^j\}$. Let us suppose against all reason that $\mu' \in \mu_\Psi^-$ exists, having the property that $\mu' \neq \mu$ and $\mu' \neq \mu \oplus \varepsilon^j$. Then $\mu'' \in \mathbf{B}^n$ and $\nu \in \mathbf{B}^n$ exist with $\Psi(\mu') = \mu''$ and $\Psi^\nu(\mu') = \mu$. Like before, the situation $\mu' = \mu''$ is impossible, because it implies $\mu' = \Psi(\mu') = \Psi^\nu(\mu') = \mu$, therefore $\mu' \neq \mu''$.

From the definition (39) of Ψ we get the existence of $k \in \{1, ..., n\}$ with $\mu'' = \mu' \oplus \varepsilon^k$, resulting furthermore that one of $\Phi^{-1}(\mu') = \{\mu' \oplus \varepsilon^k\}$, $\Phi^{-1}(\mu') = \{\mu', \mu' \oplus \varepsilon^k\}$ is true. On the other hand $\{\Psi^\nu(\mu') | \nu \in \mathbf{B}^n\} = \{\mu', \mu' \oplus \varepsilon^k\}$ and $\mu \in \{\mu', \mu' \oplus \varepsilon^k\}$. But $\mu = \mu'$ is impossible, thus $\mu = \mu' \oplus \varepsilon^k$.

Case $\Phi^{-1}(\mu') = \{\mu' \oplus \varepsilon^k\}$, i.e. $\Phi^{-1}(\mu \oplus \varepsilon^k) = \{\mu\}$

In this case $\Phi(\mu) = \mu \oplus \varepsilon^k = \mu \oplus \varepsilon^j, k = j$ and $\mu = \mu' \oplus \varepsilon^j$, contradiction.

Case $\Phi^{-1}(\mu') = \{\mu', \mu' \oplus \varepsilon^k\}$, i.e. $\Phi^{-1}(\mu \oplus \varepsilon^k) = \{\mu \oplus \varepsilon^k, \mu\}$

The fact that $\Phi(\mu) = \mu \oplus \varepsilon^k = \mu \oplus \varepsilon^j$ implies $k = j$ and $\mu = \mu' \oplus \varepsilon^j$, contradiction.

We have obtained that such a μ' does not exist.

Case iv) μ is a sink for Φ, some $i \in \{1, ..., n\}$ exists such that $\Psi(\mu) = \mu \oplus \varepsilon^i$, $\Phi^{-1}(\mu) = \{\mu, \mu \oplus \varepsilon^i\}$,

$$\mu_\Phi^- = \{\mu, \mu \oplus \varepsilon^i\} = \mu_\Psi^+,$$

$$\mu_\Phi^+ = \{\mu\}$$

and we prove that $\mu_\Psi^- = \{\mu\}$.

$\{\mu\} \subset \mu_\Psi^-$. Obvious.

$\mu_\Psi^- \subset \{\mu\}$. We suppose against all reason that $\mu' \in \mu_\Psi^-$ exists, $\mu' \neq \mu$, in other words $\mu'' \in \mathbf{B}^n$ and $\nu \in \mathbf{B}^n$ exist such that $\Psi(\mu') = \mu''$ and $\Psi^\nu(\mu') = \mu$. The situation $\mu' = \mu''$ cannot take place, because it implies the contradiction $\mu' = \Psi(\mu') = \Psi^\nu(\mu') = \mu$. As $\mu' \neq \mu''$ and taking into account (39), we obtain the existence of $j \in \{1, ..., n\}$ with $\mu'' = \mu' \oplus \varepsilon^j$. This fact shows us furthermore that one of $\Phi^{-1}(\mu') = \{\mu' \oplus \varepsilon^j\}$, $\Phi^{-1}(\mu') = \{\mu', \mu' \oplus \varepsilon^j\}$ is true. As $\{\Psi^\nu(\mu') | \nu \in \mathbf{B}^n\} = \{\mu', \mu' \oplus \varepsilon^j\}$ thus we have $\mu \in \{\mu', \mu' \oplus \varepsilon^j\}$, and because the hypothesis has excluded the case $\mu = \mu'$, we infer that $\mu = \mu' \oplus \varepsilon^j$ is the only possibility.

Case $\Phi^{-1}(\mu') = \{\mu' \oplus \varepsilon^j\}$, i.e. $\Phi^{-1}(\mu \oplus \varepsilon^j) = \{\mu\}$

The contradiction $\mu \oplus \varepsilon^j = \Phi(\mu) = \mu$ results.

Case $\Phi^{-1}(\mu') = \{\mu', \mu' \oplus \varepsilon^j\}$, i.e. $\Phi^{-1}(\mu \oplus \varepsilon^j) = \{\mu \oplus \varepsilon^j, \mu\}$

We infer $\mu \oplus \varepsilon^j = \Phi(\mu) = \mu$, representing a contradiction.

The conclusion is that such a μ' does not exist.

In all the Cases i),...,iv) we proved that $\mu_\Phi^- = \mu_\Psi^+, \mu_\Phi^+ = \mu_\Psi^-$ hold, thus the time-reversal symmetry of Φ and Ψ results.

b) At Case i) we have $\Psi^{-1}(\mu) = \{\mu, \mu \oplus \varepsilon^i\}$, see (38); at Case ii) we have $\Psi^{-1}(\mu) = \{\mu\}$, see (36); at Case iii) we have $\Psi^{-1}(\mu) = \{\mu \oplus \varepsilon^j\}$, see (37); and at Case iv) we have $\Psi^{-1}(\mu) = \emptyset$, see (35). The conclusion is that $\Delta_\Psi(u)$ is true. □

Theorem 45. *The following properties are equivalent:*

a) Φ fulfills the technical condition of proper operation and Ψ exists such that the time-reversal symmetry of Φ and Ψ is true;

b) $\forall \mu \in \mathbf{B}^n, \Delta_\Phi(\mu)$ holds.

Proof. a)$\Longrightarrow$b) We suppose against all reason that $\mu \in \mathbf{B}^n$ exists such that $\Delta_\Phi(\mu)$ is false. This means, since Φ fulfills the technical condition of proper operation, the existence of $i, j \in \{1, ..., n\}$ with $\mu \oplus \varepsilon^i, \mu \oplus \varepsilon^j \in \Phi^{-1}(\mu)$, see Theorem 28, where (26),...,(29) are false and one of (30),...,(33) is true. We denote with Ψ the unique function such that the time-reversal symmetry of Φ and Ψ holds (Theorem 26) and we have $\mu \oplus \varepsilon^i, \mu \oplus \varepsilon^j, \mu \oplus \varepsilon^i \oplus \varepsilon^j \in \mu_\Phi^- = \mu_\Psi^+$ (this is the structure of linear variety of μ_Ψ^+, see Theorem 11) thus Ψ fulfills $\Psi(\mu) = \mu \oplus \bigoplus_{k \in I} \varepsilon^k$, where $I \subset \{1, ..., n\}$ and $i, j \in I$. Ψ does not fulfill the technical condition of proper operation, contradiction with Theorem 39.

b)$\Longrightarrow$a) From b) we get that Φ fulfills the technical condition of proper operation, and Theorem 44 shows how Ψ can be defined such that the time-reversal symmetry of Φ and Ψ holds. □

$$(0,0) \longleftarrow (0,\underline{1}) \longleftarrow (\underline{1},1) \qquad\qquad (1,0)$$

Figure 12. The function Ψ such that the time-reversal symmetry of the function Φ from Figure 5 and Ψ holds.

Example 46. *The identity* $1_{\mathbf{B}^n} : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *fulfills* $\forall\mu \in \mathbf{B}^n, \Delta_{1_{\mathbf{B}^n}}(\mu)$.

Example 47. *In Figure 5 we have a function* Φ *that fulfills* $\forall\mu \in \mathbf{B}^n$, $\Delta_\Phi(\mu)$:

$$\Phi^{-1}(0,0) = \varnothing, \text{ see (35),}$$
$$\Phi^{-1}(0,1) = \{(0,0)\}, \text{ see (37),}$$
$$\Phi^{-1}(1,1) = \{(0,1),(1,1)\}, \text{ see (38),}$$
$$\Phi^{-1}(1,0) = \{(1,0)\}, \text{ see (36).}$$

The function Ψ *is defined by (39):*

$$\Psi(0,0) = (0,0), \text{ see (35),}$$
$$\Psi(0,1) = (0,0), \text{ see (37),}$$
$$\Psi(1,0) = (1,0), \text{ see (36),}$$
$$\Psi(1,1) = (0,1), \text{ see (38)}$$

and its state portrait is drawn in Figure 12. The time-reversal symmetry af Φ *and* Ψ *holds.*

Example 48. *To be noticed that the functions* Φ, Ψ *from Figure 9 a), b) fulfill* $\forall\mu \in \mathbf{B}^n, \Delta_\Phi(\mu)$ *and* $\forall\mu \in \mathbf{B}^n, \Delta_\Psi(\mu)$*, for example* $\Phi^{-1}(0,0) = \{(0,1)\}$, $\Phi^{-1}(0,1) = \{(0,0)\}$, $\Phi^{-1}(1,1) = \varnothing$ *and* $\Phi^{-1}(1,0) = \{(1,0),(1,1)\}$.

9. Conclusion

Corollary 49. *We suppose that* Φ *fulfills* $\forall\mu \in \mathbf{B}^n, \Delta_\Phi(\mu)$. *Then*

a) Φ *fulfills the technical condition of proper operation and* $\forall\mu \in \mathbf{B}^n, \forall\alpha \in \widehat{\Pi}_n$*, a sequence* $-1 = j_0 < j_1 < ... < j_k < ...$ *exists having the property that* $\forall k \in \mathbf{N}$,

a.i) $\widehat{\Phi}^\alpha(\mu, j_k) = \Phi^{(k)}(\mu)$;

a.ii) if $j_{k+1} - j_k \geq 2$*, then* $\forall j \in \{j_k, j_k+1, ..., j_{k+1}-1\}, \widehat{\Phi}^\alpha(\mu, j) = \Phi^{(k)}(\mu)$.

Let Ψ *be defined like at (39).*

b) Ψ *fulfills* $\forall\mu \in \mathbf{B}^n, \Delta_\Psi(\mu)$*, it fulfills also the technical condition of proper operation and* $\forall\mu \in \mathbf{B}^n, \forall\alpha \in \widehat{\Pi}_n$*, a sequence* $-1 = j'_0 < j'_1 < ... < j'_k < ...$ *exists having the property that* $\forall k \in \mathbf{N}$,

b.i) $\widehat{\Psi}^\alpha(\mu, j'_k) = \Psi^{(k)}(\mu)$,

b.ii) if $j'_{k+1} - j'_k \geq 2$*, then* $\forall j \in \{j'_k, j'_k+1, ..., j'_{k+1}-1\}, \widehat{\Psi}^\alpha(\mu, j) = \Psi^{(k)}(\mu)$;

c) we have

$$\forall\mu \in \mathbf{B}^n, \forall\alpha \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_{\text{-}}, \exists\beta \in \widehat{\Pi}_n, \widehat{\Phi}^\beta(\widehat{\Psi}^\alpha(\mu), k), k) = \mu,$$
$$\forall\mu \in \mathbf{B}^n, \forall\beta \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_{\text{-}}, \exists\alpha \in \widehat{\Pi}_n, \widehat{\Psi}^\alpha(\widehat{\Phi}^\beta(\mu), k), k) = \mu.$$

Proof. The fact that Ψ satisfies $\forall\mu \in \mathbf{B}^n, \Delta_\Psi(\mu)$ results from Theorem 44. Φ and Ψ satisfy the technical condition of proper operation from Remark 42 and the existence $\forall\mu \in \mathbf{B}^n, \forall\alpha \in \widehat{\Pi}_n$, of two sequences $(j_k), (j'_k)$ like in the statement of the Corollary results from Theorem 36.

Theorem 44 shows that the time-reversal symmetry of Φ and Ψ holds, in other words the statement c) of the Corollary is true too. □

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