The model of the ideal rotary element of Morita

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1 Introduction

Reversible computing is a concept reflecting physical reversibility. Until now several reversible systems such as reversible Turing machines, reversible cellular automata and reversible logic circuits have been investigated. In a series of papers Kenichi Morita defines the rotary element RE, that is a reversible logic element. By reversibility, he understands [2] that 'every computation process can be traced backward uniquely from the end to the start. In other words, they are backward deterministic systems'[2]. He shows [1] that any reversible Turing machine can be realized as a circuit composed of RE's only.

Our purpose in this paper is to use the asynchronous systems theory and the real time for the modeling of the ideal rotary element (the signal is transmitted from the input to the output without being altered and without delays, as opposed to the inertial rotary element).

2 Preliminaries

Definition 1 *The set* $\mathbf{B} = \{0,1\}$ *endowed with the usual algebraical laws* $^-$, $\cup$, $\cdot$, $\oplus$ *and with the order* $0 < 1$ *is called the* ***binary Boole algebra***.

Definition 2 *The* ***characteristic function*** $\chi_A : \mathbf{R} \to \mathbf{B}$ *of the set* $A \subset \mathbf{R}$ *is defined by* $\forall t \in A$,

$$\chi_A(t) = \begin{cases} 1, t \in A \\ 0, t \notin A \end{cases}.$$

Notation 3 *We denote by Seq the set of the sequences* $t_k \in \mathbf{R}$, $k \in \mathbf{N}$ *which are strictly increasing* $t_0 < t_1 < t_2 < ...$ *and unbounded from above. The elements of Seq will be denoted in general by* (t_k).

Definition 4 *The **signals** (or the $n-$**signals**) are by definition the $\mathbf{R} \to \mathbf{B}^n$ functions of the form*

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... \quad (1)$$

where $t \in \mathbf{R}$, $\mu \in \mathbf{B}^n$ and $(t_k) \in Seq$. The set of the signals is denoted by $S^{(n)}$.

Definition 5 *In (1), μ is called the **initial value** of x and its usual notation is $x(-\infty + 0)$.*

Definition 6 *If x fulfills (1), the $\mathbf{R} \to \mathbf{B}^n$ function*

$$x(t-0) = \mu \cdot \chi_{(-\infty,t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0,t_1]}(t) \oplus ... \oplus x(t_k) \cdot \chi_{(t_k,t_{k+1}]}(t) \oplus ...$$

*is called the **left limit** of x.*

Definition 7 *For x like previously, the $\mathbf{R} \to \mathbf{B}$ functions $\overline{x_i(t-0)}x_i(t)$, $x_i(t-0)\overline{x_i(t)}$ are called the **left semi-derivatives** of $x_i, i = \overline{1,n}$.*

Definition 8 *An **asynchronous system** is a multi-valued function $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$. U is called the **input set** and its elements $u \in U$ are called (**admissible**) **inputs**, while the functions $x \in f(u)$ are called (**possible**) **states**.*

3 The informal definition of the rotary element of Morita

Definition 9 *(informal) The **rotary element** RE, whose symbol is given in Figure 1, has four inputs u_1, u_2, u_3, u_4, a state x_0 and four outputs x_1, x_2, x_3, x_4. Its work has been intuitively explained by the existence of a 'rotating bar', see Figures 2 and 3. If (Figure 2) the state x_0 is in the horizontal position, symbolized by us with $x_0(t-0) = 0$, then $u_1(t) = 1$ -this was indicated with a bullet- makes the state remain horizontal $x_0(t) = 0$ and the bullet be transmitted horizontally to x_1, thus $x_1(t) = 1$. If (Figure 3) x_0 is in the vertical position, symbolized by us with $x_0(t-0) = 1$ and if $u_1(t) = 1$, then the state x_0 rotates counterclockwise, i.e. it switches from 1 to $0 : x_0(t) = 0$ and the bullet is transmitted to $x_4 : x_4(t) = 1$. No two distinct inputs may be activated at a time -i.e. at most one bullet exists- moreover, between the successive activation of the inputs, some time interval must exist when all the inputs are null. If all the inputs are null, $u_1(t) = u_2(t) = u_3(t) = u_4(t) = 0$ -i.e. if no bullet exists- then x_0 keeps its previous value, $x_0(t) = x_0(t-0)$ and $x_1(t) = x_2(t) = x_3(t) = x_4(t) = 0$. The definition of the rotary element is completed by requests of symmetry.*

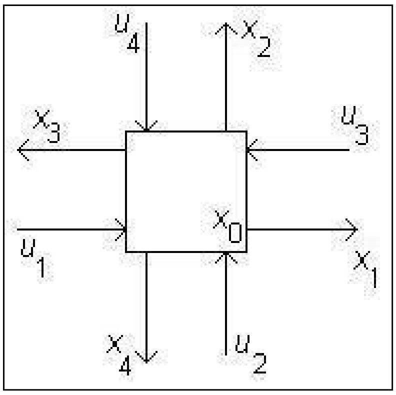

Figure 1: The symbol of the rotary element

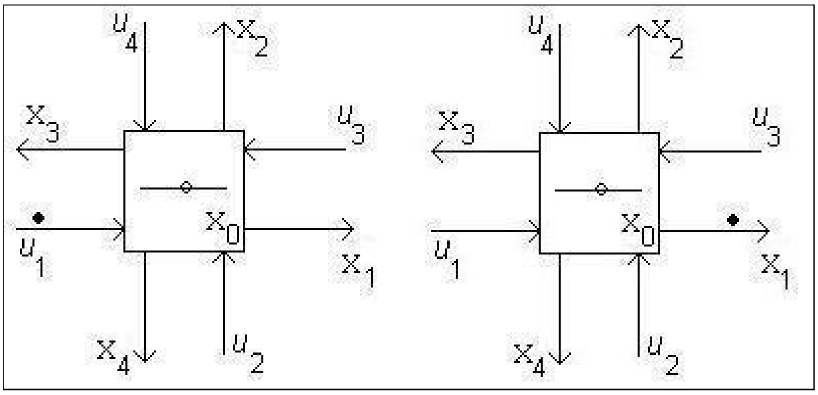

Figure 2: RE in state $x_0(t-0) = 0$ and with the input $u_1(t) = 1$ computes $x_0(t) = 0$ and $x_1(t) = 1$

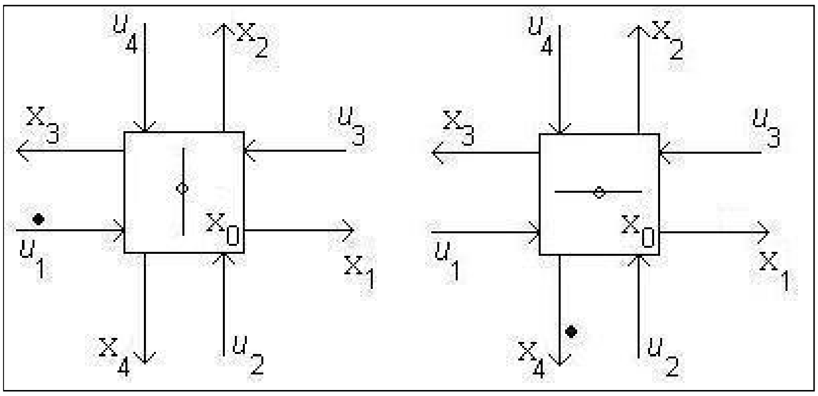

Figure 3: RE in state $x_0(t-0) = 1$ and with the input $u_1(t) = 1$ computes $x_0(t) = 0$ and $x_4(t) = 1$

Remark 10 *Morita states the 'reversibility' of RE. This means that in Figures 2 and 3 where time passes from the left to the right we may say looking at the right picture which the left picture is. In other words, knowing the position of the rotating bar and the values of the outputs allows us to know the previous position of the rotating bar and the values of the inputs. In this 'reversed' manner of interpreting things the state* x_0 *rotates clockwise,* $x_1, ..., x_4$ *become inputs and* $u_1, ..., u_4$ *become outputs.*

4 Systems of equations

Theorem 11 *Let be the functions* $f', f'' : \mathbf{R} \to \mathbf{B}$. *The following statements a), b) are equivalent:*

a) one of a.1),...,a.5) holds

a.1) $\forall t \in \mathbf{R}$,

$$f'(t) = 0,$$
$$f''(t) = 0;$$

a.2) $\exists t_0 \in \mathbf{R}, \forall t \in \mathbf{R}$,

$$f'(t) = \chi_{\{t_0\}}(t),$$
$$f''(t) = 0;$$

a.3) $\exists t_0 \in \mathbf{R}, \exists t_1 \in \mathbf{R}, t_0 < t_1, \forall t \in \mathbf{R}$,

$$f'(t) = \chi_{\{t_0\}}(t),$$
$$f''(t) = \chi_{\{t_1\}}(t);$$

a.4) $\exists k > 1, \exists t_0 \in \mathbf{R}, ..., \exists t_k \in \mathbf{R}, t_0 < ... < t_k, \forall t \in \mathbf{R}$,

$$f'(t) = f'(t_0) \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus f'(t_k) \cdot \chi_{\{t_k\}}(t),$$
$$f''(t) = f''(t_0) \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus f''(t_k) \cdot \chi_{\{t_k\}}(t),$$

$\forall k' \in \{0, ..., k\}$,

$$f'(t_{k'}) = \begin{cases} 1, k' = even, \\ 0, k' = odd \end{cases}, f''(t_{k'}) = \begin{cases} 0, k' = even, \\ 1, k' = odd \end{cases};$$

a.5) $\exists (t_k) \in Seq, \forall t \in \mathbf{R}$,

$$f'(t) = f'(t_0) \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus f'(t_k) \cdot \chi_{\{t_k\}}(t) \oplus ...,$$
$$f''(t) = f''(t_0) \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus f''(t_k) \cdot \chi_{\{t_k\}}(t) \oplus ...,$$

$\forall k \in \mathbf{N}$,

$$f'(t_k) = \begin{cases} 1, k = even, \\ 0, k = odd \end{cases}, f''(t_k) = \begin{cases} 0, k = even, \\ 1, k = odd \end{cases};$$

b) the system of equations

$$\begin{cases} \overline{w(t-0)}w(t) = f'(t), \\ w(t-0)\overline{w(t)} = f''(t) \end{cases} \tag{2}$$

has a solution $w \in S^{(1)}$ *with* $w(-\infty + 0) = 0$.

Remark 12 *We have a similar statement with the one of Theorem 11, which is obtained by the replacement in a) of 'even' with 'odd' and the replacement in b) of* $w(-\infty + 0) = 0$ *with* $w(-\infty + 0) = 1$. *This is the dual of Theorem 11.*

Theorem 13 *If one of a), b) from Theorem 11 is true, then the solution of (2) is unique and is given by*

$$w(t) = \overline{w(t-0)}f'(t) \cup w(t-0)\overline{f''(t)}, \tag{3}$$

where $w(-\infty+0) = 0$. *If one of a), b) from the dual of Theorem 11 holds, then the solution of (2) is unique and is given by (3), where* $w(-\infty+0) = 1$.

Theorem 14 *For any* $f', f'' \in S^{(1)}$, *if* $f'(-\infty + 0) = f''(-\infty + 0) = 0$, *then* $\forall t \in \mathbf{R}$,

$$f'(t)f''(t) = 0 \tag{4}$$

if and only if the system of equations

$$\begin{cases} \overline{w(t-0)}w(t) = \overline{w(t-0)}f'(t), \\ w(t-0)\overline{w(t)} = w(t-0)f''(t) \end{cases} \tag{5}$$

has a solution $w \in S$. *If (5) has a solution* w, *this solution is unique and is given by*

$$w(t) = \overline{w(t-0)}f'(t) \cup w(t-0)\overline{f''(t)}. \tag{6}$$

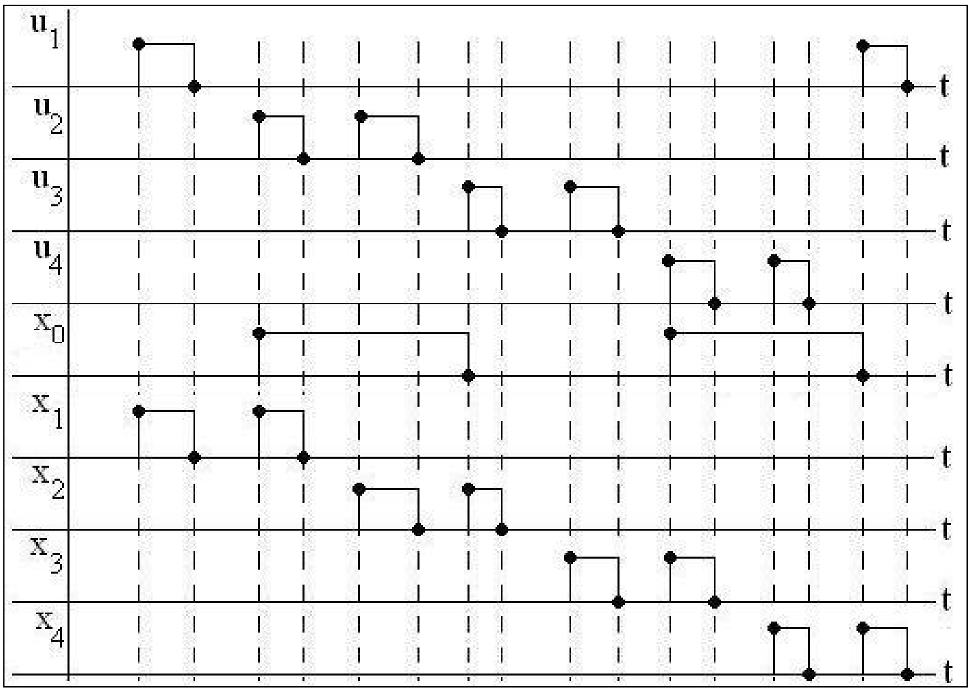

Figure 4: The ideal RE

5 The ideal RE

Remark 15 *We get from Section 3 that RE may be described by the following table*

$x_0(t-0)$	$u_1(t)$	$u_2(t)$	$u_3(t)$	$u_4(t)$	$x_0(t)$	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0
0	0	1	0	0	1	1	0	0	0
0	0	0	1	0	0	0	0	1	0
0	0	0	0	1	1	0	0	1	0
1	0	0	0	0	1	0	0	0	0
1	1	0	0	0	0	0	0	0	1
1	0	1	0	0	1	0	1	0	0
1	0	0	1	0	0	0	1	0	0
1	0	0	0	1	1	0	0	0	1

Table 1

and by Figure 4. We have asked that all the variables belong to $S^{(1)}$ and that any switch of the input is transmitted to the state x_0 and to the outputs x_1, x_2, x_3, x_4 instantly, without being altered and without delays.

This approximation is called by us in the following 'the ideal RE', as opposed to 'the inertial RE'.

We shall suppose the way that we always use to that the outputs are states, thus the state vector in this Chapter will have the coordinates $x = (x_0, x_1, x_2, x_3, x_4) \in S^{(5)}$.

Notation 16 *We denote*

$$\mathbf{0} = (0,0,0,0) \in \mathbf{B}^4.$$

Notation 17 *We denote by* $D, D^* \subset \mathbf{B}^4$ *the sets*

$$D = \{\mathbf{0}, (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\},$$

$$D^* = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}.$$

Definition 18 *We define the **set of the admissible inputs*** $U \in P^*(S^{(4)})$ *by*

$$U = \{\mathbf{0}\} \cup \{\lambda \cdot \chi_{[t_0,\infty)} | t_0 \in \mathbf{R}, \lambda \in D^*\}$$
$$\cup \{\lambda^0 \cdot \chi_{[t_0,t_1)} \oplus \lambda^1 \cdot \chi_{[t_2,t_3)} \oplus ... \oplus \lambda^k \cdot \chi_{[t_{2k},t_{2k+1})} |$$
$$k \in \mathbf{N}, t_0, ..., t_{2k+1} \in \mathbf{R}, t_0 < ... < t_{2k+1}, \lambda^0, ..., \lambda^k \in D^*\}$$
$$\cup \{\lambda^0 \cdot \chi_{[t_0,t_1)} \oplus \lambda^1 \cdot \chi_{[t_2,t_3)} \oplus ... \oplus \lambda^k \cdot \chi_{[t_{2k},\infty)} |$$
$$k \in \mathbf{N}, t_0, ..., t_{2k} \in \mathbf{R}, t_0 < ... < t_{2k}, \lambda^0, ..., \lambda^k \in D^*\}$$
$$\cup \{\lambda^0 \cdot \chi_{[t_0,t_1)} \oplus \lambda^1 \cdot \chi_{[t_2,t_3)} \oplus ... \oplus \lambda^k \cdot \chi_{[t_{2k},t_{2k+1})} \oplus ... | (t_k) \in Seq, \lambda^k \in D^*, k \in \mathbf{N}\}.$$

Theorem 19 *The functions* $u \in U$ *fulfill*

a) $u(-\infty + 0) = \mathbf{0}$;

b) $\forall i \in \{1, ..., 4\}, \forall j \in \{1, ..., 4\}, i \neq j$ *implies*

$$u_i(t) u_j(t) = 0, \tag{7}$$

$$\overline{u_i(t-0)} u_i(t) u_j(t-0) \overline{u_j(t)} = 0; \tag{8}$$

c) $\forall u \in U$, $\forall d \in \mathbf{R}$, $u \circ \tau^d \in U$.

Remark 20 *The set* U *is the set of the inputs, as given in the informal Definition 9. Thus, the equation (7) reproduces the request 'no two distinct inputs may be activated at a time' from that Definition. The equation (8) shows that the end of a 1-pulse on an input* u_j *cannot coincide with the beginning of a 1-puse on another input* u_i*, these two events must be separated in time. This property corresponds to the statement from the informal Definition 9 that 'between the successive activation of the inputs, some time interval must exist when all the inputs are null'.*

Definition 21 *We define the* ***set of the initial (values of the) states*** *by*

$$\Theta_0 = \{(0,0,0,0,0),(1,0,0,0,0)\}.$$

Theorem 22 *Let $u \in U$ be given.*

a) The equations

$$\begin{cases} \overline{x_0(t-0)}x_0(t) = \overline{x_0(t-0)}(u_2(t) \cup u_4(t)) \\ x_0(t-0)\overline{x_0(t)} = x_0(t-0)(u_1(t) \cup u_3(t)) \end{cases}, \tag{9}$$

$$x_0(t) = \overline{x_0(t-0)}(u_2(t) \cup u_4(t)) \cup x_0(t-0)\,\overline{u_1(t)}\,\overline{u_3(t)} \tag{10}$$

that refer to the variables from Table 1 and Figure 4 have the same unique solution $x_0 \in S$ whenever the initial condition $x_0(-\infty+0) \in \mathbf{B}$ is indicated.

b) The equations

$$\begin{cases} \overline{x_1(t-0)}x_1(t) = \overline{x_0(t-0)}\,\overline{u_1(t-0)}u_1(t) \cup \overline{x_0(t-0)}\,\overline{u_2(t-0)}u_2(t) \\ x_1(t-0)\overline{x_1(t)} = \overline{x_0(t-0)}u_1(t-0)\overline{u_1(t)} \cup x_0(t-0)u_2(t-0)\overline{u_2(t)} \end{cases} \tag{11}$$

$$x_1(t) = \overline{x_1(t-0)}\,\overline{x_0(t-0)}(\overline{u_1(t-0)}u_1(t) \cup \overline{u_2(t-0)}u_2(t)) \cup \tag{12}$$
$$\cup x_1(t-0)(x_0(t-0) \cup \overline{u_1(t-0)} \cup u_1(t))(\overline{x_0(t-0)} \cup \overline{u_2(t-0)} \cup u_2(t))$$

that refer to the variables from Table 1 and Figure 4 have the same unique solution $x_1 \in S$, whenever $x_1(-\infty+0) = 0$; similar statements hold referring to x_2, x_3, x_4.

Definition 23 *We consider the functions $u \in U$ and $x \in S^{(5)}$, $x(-\infty+0) \in \Theta_0$. The equations*

$$x_0(t) = \overline{x_0(t-0)}(u_2(t) \cup u_4(t)) \cup x_0(t-0)\,\overline{u_1(t)}\,\overline{u_3(t)}, \tag{13}$$

$$x_1(t) = \overline{x_1(t-0)}\,\overline{x_0(t-0)}(\overline{u_1(t-0)}u_1(t) \cup \overline{u_2(t-0)}u_2(t)) \cup \tag{14}$$
$$\cup x_1(t-0)(x_0(t-0) \cup \overline{u_1(t-0)} \cup u_1(t))(\overline{x_0(t-0)} \cup \overline{u_2(t-0)} \cup u_2(t)),$$

$$x_2(t) = \overline{x_2(t-0)}\,x_0(t-0)(\overline{u_2(t-0)}u_2(t) \cup \overline{u_3(t-0)}u_3(t)) \cup \tag{15}$$
$$\cup x_2(t-0)(\overline{x_0(t-0)} \cup \overline{u_2(t-0)} \cup u_2(t))(x_0(t-0) \cup \overline{u_3(t-0)} \cup u_3(t)),$$

$$x_3(t) = \overline{x_3(t-0)}\,\overline{x_0(t-0)}(\overline{u_3(t-0)}u_3(t) \cup \overline{u_4(t-0)}u_4(t)) \cup \tag{16}$$
$$\cup x_3(t-0)(x_0(t-0) \cup \overline{u_3(t-0)} \cup u_3(t))(\overline{x_0(t-0)} \cup \overline{u_4(t-0)} \cup u_4(t)),$$

$$x_4(t) = \overline{x_4(t-0)}\,x_0(t-0)(\overline{u_4(t-0)}u_4(t) \cup \overline{u_1(t-0)}u_1(t)) \cup \tag{17}$$
$$\cup x_4(t-0)(\overline{x_0(t-0)} \cup \overline{u_4(t-0)} \cup u_4(t))(x_0(t-0) \cup \overline{u_1(t-0)} \cup u_1(t))$$

are called the ***equations of the ideal RE*** *(of Morita) and the system $f : U \to P^*(S^{(5)})$ that is defined by them is called the* ***ideal RE*** *(of Morita); (13),...,(17) are called the* ***state equations*** *of f.*

Remark 24 *The system f is finite, since $\forall u \in U, f(u)$ has two elements $\{x, x'\}$ satisfying $x(-\infty+0) = (0,0,0,0,0)$ and $x'(-\infty+0) = (1,0,0,0,0)$.*

Notation 25 *Let be $\mu \in \Theta_0$. We denote by $f_\mu : U \to S^{(5)}$ the deterministic system $\forall u \in U$,*

$$f_\mu(u) = x$$

where x fulfills $x(-\infty+0) = \mu$ and (13),...,(17).

6 The analysis of the ideal RE

Definition 26 *We consider the function $\Phi : \mathbf{B}^5 \times \mathbf{B}^4 \to \mathbf{B}^5$ defined by $\forall(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4) \in \mathbf{B}^5$, $\forall(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbf{B}^4$,*

$$\Phi_0(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \overline{\mu_0}(\lambda_2 \cup \lambda_4) \cup \mu_0 \overline{\lambda_1}\,\overline{\lambda_3}, \tag{18}$$

$$\Phi_1(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \overline{\mu_0}(\lambda_1 \cup \lambda_2), \tag{19}$$

$$\Phi_2(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \mu_0(\lambda_2 \cup \lambda_3), \tag{20}$$

$$\Phi_3(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \overline{\mu_0}(\lambda_3 \cup \lambda_4), \tag{21}$$

$$\Phi_4(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \mu_0(\lambda_4 \cup \lambda_1). \tag{22}$$

Remark 27 *The function Φ makes true the following Table*

μ_0	λ_1	λ_2	λ_3	λ_4	$\Phi_0(\mu,\lambda)$	$\Phi_1(\mu,\lambda)$	$\Phi_2(\mu,\lambda)$	$\Phi_3(\mu,\lambda)$	$\Phi_4(\mu,\lambda)$
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0
0	0	1	0	0	1	1	0	0	0
0	0	0	1	0	0	0	0	1	0
0	0	0	0	1	1	0	0	1	0
1	0	0	0	0	1	0	0	0	0
1	1	0	0	0	0	0	0	0	1
1	0	1	0	0	1	0	1	0	0
1	0	0	1	0	0	0	1	0	0
1	0	0	0	1	1	0	0	0	1

Table 2

that this coincides with Table 1, where we have obviously put $\mu_0 = x_0(t-0)$, $\lambda_1 = u_1(t), ..., \lambda_4 = u_4(t)$, $\mu = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4) \in \mathbf{B}^5$, $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbf{B}^4$, $\Phi_0(\mu, \lambda) = x_0(t), ..., \Phi_4(\mu, \lambda) = x_4(t)$.

Notation 28 *For all $k \in \mathbf{N}$, $\lambda^0, ..., \lambda^k, \lambda^{k+1} \in D$ and for any $\mu \in \Theta_0$, the vectors $\Phi(\mu, \lambda^0...\lambda^k\lambda^{k+1}) \in \mathbf{B}^5$ are iteratively defined by*

$$\Phi(\mu, \lambda^0...\lambda^k\lambda^{k+1}) = \Phi(\Phi(\mu, \lambda^0...\lambda^k), \lambda^{k+1}).$$

Remark 29 *The iterates $\Phi(\mu,\lambda^0...\lambda^k)$ show how Φ acts when a succession of input values $\lambda^0,...,\lambda^k \in D^*$ is applied in the initial state $\mu \in \Theta_0$. For example we have*

$$\Phi(\mu,\mathbf{0}) = \mu,$$
$$\Phi(\mu,\lambda\mathbf{0}\lambda') = \Phi(\mu,\lambda\lambda')$$

for any $\mu \in \Theta_0$ and $\lambda,\lambda' \in D^*$.

Theorem 30 *When $\mu \in \Theta_0$, $\lambda,\lambda^0,...,\lambda^k,... \in D^*$ and $(t_k) \in Seq$, the following statements are true:*

$$f_\mu(\mathbf{0}) = \mu, \tag{23}$$

$$f_\mu(\lambda \cdot \chi_{[t_0,\infty)}) = \mu \cdot \chi_{(-\infty,t_0)} \oplus \Phi(\mu,\lambda) \cdot \chi_{[t_0,\infty)}, \tag{24}$$

$$f_\mu(\lambda^0 \cdot \chi_{[t_0,t_1)} \oplus \lambda^1 \cdot \chi_{[t_2,t_3)} \oplus ... \oplus \lambda^k \cdot \chi_{[t_{2k},t_{2k+1})}) = \tag{25}$$

$$= \mu \cdot \chi_{(-\infty,t_0)} \oplus \Phi(\mu,\lambda^0) \cdot \chi_{[t_0,t_1)} \oplus \Phi(\mu,\lambda^0\mathbf{0}) \cdot \chi_{[t_1,t_2)} \oplus \Phi(\mu,\lambda^0\lambda^1) \cdot \chi_{[t_2,t_3)} \oplus ...$$
$$... \oplus \Phi(\mu,\lambda^0...\lambda^{k-1}\mathbf{0}) \cdot \chi_{[t_{2k-1},t_{2k})} \oplus \Phi(\mu,\lambda^0...\lambda^k) \cdot \chi_{[t_{2k},t_{2k+1})},$$

$$f_\mu(\lambda^0 \cdot \chi_{[t_0,t_1)} \oplus \lambda^1 \cdot \chi_{[t_2,t_3)} \oplus ... \oplus \lambda^k \cdot \chi_{[t_{2k},\infty)}) = \tag{26}$$

$$= \mu \cdot \chi_{(-\infty,t_0)} \oplus \Phi(\mu,\lambda^0) \cdot \chi_{[t_0,t_1)} \oplus \Phi(\mu,\lambda^0\mathbf{0}) \cdot \chi_{[t_1,t_2)} \oplus \Phi(\mu,\lambda^0\lambda^1) \cdot \chi_{[t_2,t_3)} \oplus ...$$
$$... \oplus \Phi(\mu,\lambda^0...\lambda^{k-1}\mathbf{0}) \cdot \chi_{[t_{2k-1},t_{2k})} \oplus \Phi(\mu,\lambda^0...\lambda^k) \cdot \chi_{[t_{2k},\infty)},$$

$$f_\mu(\lambda^0 \cdot \chi_{[t_0,t_1)} \oplus \lambda^1 \cdot \chi_{[t_2,t_3)} \oplus ... \oplus \lambda^k \cdot \chi_{[t_{2k},t_{2k+1})} \oplus ...) = \tag{27}$$

$$= \mu \cdot \chi_{(-\infty,t_0)} \oplus \Phi(\mu,\lambda^0) \cdot \chi_{[t_0,t_1)} \oplus \Phi(\mu,\lambda^0\mathbf{0}) \cdot \chi_{[t_1,t_2)} \oplus \Phi(\mu,\lambda^0\lambda^1) \cdot \chi_{[t_2,t_3)} \oplus ...$$
$$... \oplus \Phi(\mu,\lambda^0...\lambda^{k-1}\mathbf{0}) \cdot \chi_{[t_{2k-1},t_{2k})} \oplus \Phi(\mu,\lambda^0...\lambda^k) \cdot \chi_{[t_{2k},t_{2k+1})} \oplus ...$$

Theorem 31 $\forall \mu \in \Theta_0, \forall u \in U, f_\mu(u) \in S^{(1)} \times U.$

Theorem 32 *a)* $\forall \mu \in \Theta_0, \forall \mu' \in \Theta_0, \forall u \in U,$

$$\mu \neq \mu' \Longrightarrow f_\mu(u) \neq f_{\mu'}(u);$$

b) $\forall \mu \in \Theta_0, \forall u \in U, \forall u' \in U,$

$$u \neq u' \Longrightarrow f_\mu(u) \neq f_\mu(u');$$

c) $\forall u \in U, \forall u' \in U,$

$$u \neq u' \Longrightarrow f(u) \cap f(u') = \emptyset.$$

Remark 33 *The previous Theorem states some injectivity properties of f. The surjectivity property*

$$\forall x \in S \times U, \exists \mu \in \Theta_0, \exists u \in U, f_\mu(u) = x$$

is not true, since if we take for arbitrary $t_0 \in \mathbf{R}$

$$x(t) = (0,0,1,0,0) \cdot \chi_{[t_0,\infty)}(t),$$

we get that $\forall \mu \in \Theta_0, \forall u \in U, f_\mu(u) \neq x$.

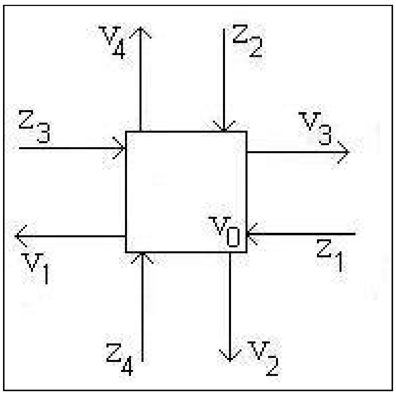

Figure 5: The symbol of the reversed RE

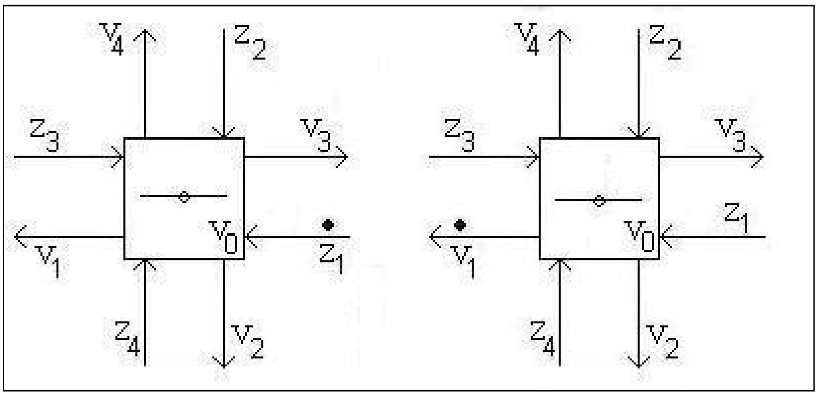

Figure 6: The reversed RE in state $v_0(t-0) = 0$ and $z_1(t) = 1$ computes $v_0(t) = 0$ and $v_1(t) = 1$

7 The reversed ideal RE

Remark 34 *The symbol of the reversed RE is given in Figure 5, to be compared with Figure 1 and a clockwise rotation of the rotating bar takes place. We can reverse Figures 2, 3 and obtain Figures 6, 7. We get*

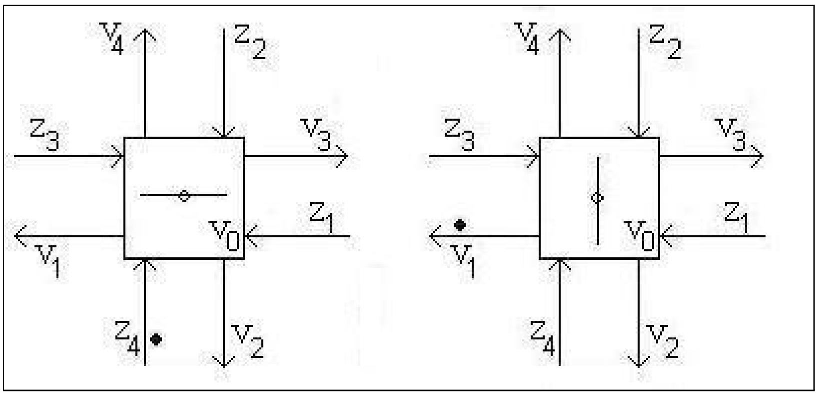

Figure 7: The reversed RE in state $v_0(t-0)=0$ and $z_4(t)=1$ computes $v_0(t)=1$ and $v_1(t)=1$

Table 3 instead of Table 1

$v_0(t-0)$	$z_1(t)$	$z_2(t)$	$z_3(t)$	$z_4(t)$	$v_0(t)$	$v_1(t)$	$v_2(t)$	$v_3(t)$	$v_4(t)$
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0
0	0	1	0	0	1	0	0	1	0
0	0	0	1	0	0	0	0	1	0
0	0	0	0	1	1	1	0	0	0
1	0	0	0	0	1	0	0	0	0
1	1	0	0	0	0	0	1	0	0
1	0	1	0	0	1	0	1	0	0
1	0	0	1	0	0	0	0	0	1
1	0	0	0	1	1	0	0	0	1

Table 3

and Figure 8 instead of Figure 4. The coordinates of the state $v \in S^{(5)}$ are $v = (v_0, v_1, v_2, v_3, v_4)$.

In the characterization of the reversed RE we can write equations that are similar with the equations of RE and they obey the same Theorems 11, 13 and 14.

Definition 35 *The following equations*

$$v_0(t) = \overline{v_0(t-0)}(z_2(t) \cup z_4(t)) \cup v_0(t-0)\,\overline{z_1(t)}\,\overline{z_3(t)}, \tag{28}$$

$$v_1(t) = \overline{v_1(t-0)}\,\overline{v_0(t-0)}(\overline{z_4(t-0)}z_4(t) \cup \overline{z_1(t-0)}z_1(t)) \cup \tag{29}$$

$$\cup v_1(t-0)(\overline{v_0(t-0)} \cup \overline{z_4(t-0)} \cup z_4(t))(v_0(t-0) \cup \overline{z_1(t-0)} \cup z_1(t)),$$

$$v_2(t) = \overline{v_2(t-0)}\,v_0(t-0)(\overline{z_1(t-0)}z_1(t) \cup \overline{z_2(t-0)}z_2(t)) \cup \tag{30}$$

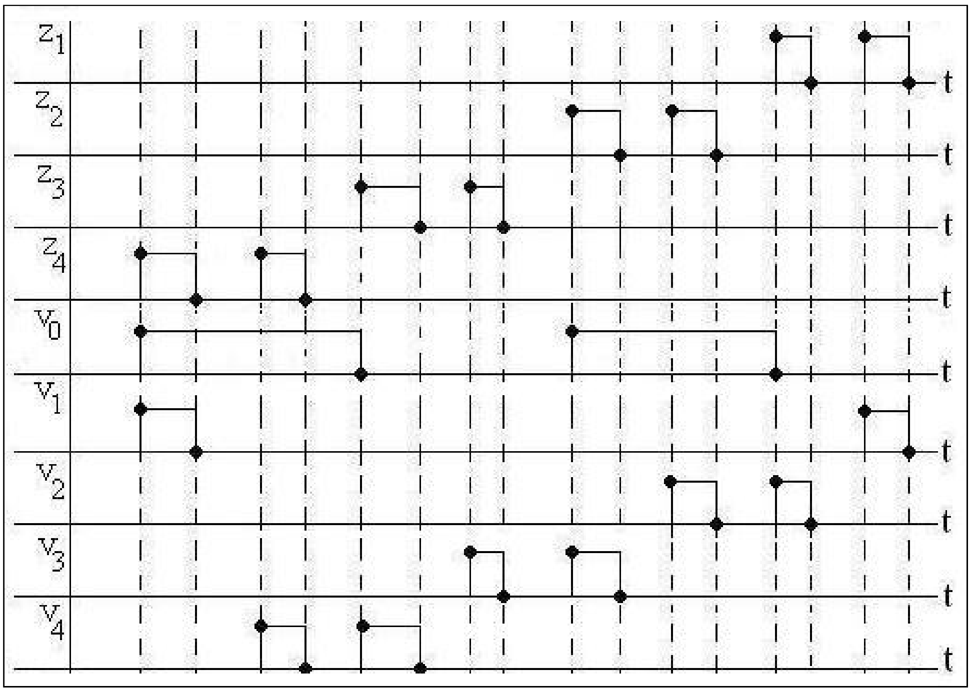

Figure 8: The ideal reversed RE

$$\cup v_2(t-0)(v_0(t-0)\cup\overline{z_1(t-0)}\cup z_1(t))(\overline{v_0(t-0)}\cup\overline{z_2(t-0)}\cup z_2(t)),$$

$$v_3(t)=\overline{v_3(t-0)}\;\overline{v_0(t-0)}(\overline{z_2(t-0)}z_2(t)\cup\overline{z_3(t-0)}z_3(t))\cup \tag{31}$$

$$\cup v_3(t-0)(\overline{v_0(t-0)}\cup\overline{z_2(t-0)}\cup z_2(t))(v_0(t-0)\cup\overline{z_3(t-0)}\cup z_3(t)),$$

$$v_4(t)=\overline{v_4(t-0)}\;v_0(t-0)(\overline{z_3(t-0)}z_3(t)\cup\overline{z_4(t-0)}z_4(t))\cup \tag{32}$$

$$\cup v_4(t-0)(v_0(t-0)\cup\overline{z_3(t-0)}\cup z_3(t))(\overline{v_0(t-0)}\cup\overline{z_4(t-0)}\cup z_4(t))$$

where $z\in U$ and $v\in S^{(5)}, v(-\infty+0)\in\Theta_0$ are called the ***equations of the reversed ideal RE****. The system $f^{-1}:U\rightarrow P^*(S^{(5)})$ that is defined by them is called the* ***reversed ideal RE****. We use to say that (28),...,(32) are the* ***state equations*** *of f^{-1}.*

Remark 36 *$\forall z\in U$, the set $f^{-1}(z)$ has two elements, corresponding to the two initial values $\nu\in\Theta_0$.*

Notation 37 *For any $\nu\in\Theta_0$, we denote by $f_\nu^{-1}:U\rightarrow S^{(5)}$ the deterministic system $\forall z\in U$,*

$$f_\nu(z)=v$$

where v fulfills (28),...,(32).

8 The analysis of the reversed ideal RE

Definition 38 *Let be the function* $\Phi^{-1} : \mathbf{B}^5 \times \mathbf{B}^4 \to \mathbf{B}^5$ *defined by* $\forall(\nu_0,\nu_1,\nu_2,\nu_3,\nu_4) \in \mathbf{B}^5$, $\forall(\delta_1,\delta_2,\delta_3,\delta_4) \in \mathbf{B}^4$,

$$\Phi_0^{-1}(\nu_0,\nu_1,\nu_2,\nu_3,\nu_4,\delta_1,\delta_2,\delta_3,\delta_4) = \overline{\nu_0}(\delta_2 \cup \delta_4) \cup \nu_0\overline{\delta_1}\,\overline{\delta_3}, \tag{33}$$

$$\Phi_1^{-1}(\nu_0,\nu_1,\nu_2,\nu_3,\nu_4,\delta_1,\delta_2,\delta_3,\delta_4) = \overline{\nu_0}(\delta_4 \cup \delta_1), \tag{34}$$

$$\Phi_2^{-1}(\nu_0,\nu_1,\nu_2,\nu_3,\nu_4,\delta_1,\delta_2,\delta_3,\delta_4) = \nu_0(\delta_1 \cup \delta_2), \tag{35}$$

$$\Phi_3^{-1}(\nu_0,\nu_1,\nu_2,\nu_3,\nu_4,\delta_1,\delta_2,\delta_3,\delta_4) = \overline{\nu_0}(\delta_2 \cup \delta_3), \tag{36}$$

$$\Phi_4^{-1}(\nu_0,\nu_1,\nu_2,\nu_3,\nu_4,\delta_1,\delta_2,\delta_3,\delta_4) = \nu_0(\delta_3 \cup \delta_4). \tag{37}$$

Remark 39 *The function* Φ^{-1} *makes true the following Table*

ν_0	δ_1	δ_2	δ_3	δ_4	$\Phi_0^{-1}(\nu,\delta)$	$\Phi_1^{-1}(\nu,\delta)$	$\Phi_2^{-1}(\nu,\delta)$	$\Phi_3^{-1}(\nu,\delta)$	$\Phi_4^{-1}(\nu,\delta)$
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0
0	0	1	0	0	1	0	0	1	0
0	0	0	1	0	0	0	0	1	0
0	0	0	0	1	1	1	0	0	0
1	0	0	0	0	1	0	0	0	0
1	1	0	0	0	0	0	1	0	0
1	0	1	0	0	1	0	1	0	0
1	0	0	1	0	0	0	0	0	1
1	0	0	0	1	1	0	0	0	1

Table 4

that coincides with Table 3, in which we have put $\nu_0 = v_0(t-0)$, $\delta_1 = z_1(t)$, ..., $\delta_4 = z_4(t)$, $\nu = (\nu_0,\nu_1,\nu_2,\nu_3,\nu_4) \in \mathbf{B}^5$, $\delta = (\delta_1,\delta_2,\delta_3,\delta_4) \in \mathbf{B}^4$, $\Phi_0^{-1}(\nu,\delta) = v_0(t)$, ..., $\Phi_4^{-1}(\nu,\delta) = v_4(t)$.

Remark 40 *The properties of* f^{-1} *coincide with those of* f, *as described in Section 6, by Theorems 30,...,32. For example similarly with Theorem 30 we have that for any* $\nu \in \Theta_0$, $\delta, \delta^0, ..., \delta^k, ... \in D^*$ *and* $(t_k) \in Seq$, *the following statements are true*

$$f_\nu^{-1}(\mathbf{0}) = \nu, \tag{38}$$

$$f_\nu^{-1}(\delta \cdot \chi_{[t_0,\infty)}) = \nu \cdot \chi_{(-\infty,t_0)} \oplus \Phi^{-1}(\nu,\delta) \cdot \chi_{[t_0,\infty)}, \tag{39}$$

$$f_\nu^{-1}(\delta^0 \cdot \chi_{[t_0,t_1)} \oplus \delta^1 \cdot \chi_{[t_2,t_3)} \oplus ... \oplus \delta^k \cdot \chi_{[t_{2k},t_{2k+1})}) = \tag{40}$$

$$= \nu \cdot \chi_{(-\infty,t_0)} \oplus \Phi^{-1}(\nu,\delta^0) \cdot \chi_{[t_0,t_1)} \oplus \Phi^{-1}(\nu,\delta^0\mathbf{0}) \cdot \chi_{[t_1,t_2)} \oplus \Phi^{-1}(\nu,\delta^0\delta^1) \cdot \chi_{[t_2,t_3)} \oplus ...$$

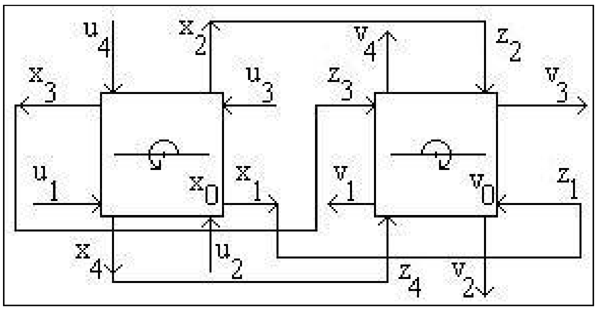

Figure 9: The study of $f^{-1} \circ f$

$$... \oplus \Phi^{-1}(\nu, \delta^0...\delta^{k-1}\mathbf{0}) \cdot \chi_{[t_{2k-1}, t_{2k})} \oplus \Phi^{-1}(\nu, \delta^0...\delta^k) \cdot \chi_{[t_{2k}, t_{2k+1})},$$

$$f_\nu^{-1}(\delta^0 \cdot \chi_{[t_0,t_1)} \oplus \delta^1 \cdot \chi_{[t_2,t_3)} \oplus ... \oplus \delta^k \cdot \chi_{[t_{2k},\infty)}) = \tag{41}$$

$$= \nu \cdot \chi_{(-\infty,t_0)} \oplus \Phi^{-1}(\nu, \delta^0) \cdot \chi_{[t_0,t_1)} \oplus \Phi^{-1}(\nu, \delta^0\mathbf{0}) \cdot \chi_{[t_1,t_2)} \oplus \Phi^{-1}(\nu, \delta^0\delta^1) \cdot \chi_{[t_2,t_3)} \oplus ...$$

$$... \oplus \Phi^{-1}(\nu, \delta^0...\delta^{k-1}\mathbf{0}) \cdot \chi_{[t_{2k-1}, t_{2k})} \oplus \Phi^{-1}(\nu, \delta^0...\delta^k) \cdot \chi_{[t_{2k},\infty)},$$

$$f_\nu^{-1}(\delta^0 \cdot \chi_{[t_0,t_1)} \oplus \delta^1 \cdot \chi_{[t_2,t_3)} \oplus ... \oplus \delta^k \cdot \chi_{[t_{2k},t_{2k+1})} \oplus ...) = \tag{42}$$

$$= \nu \cdot \chi_{(-\infty,t_0)} \oplus \Phi^{-1}(\nu, \delta^0) \cdot \chi_{[t_0,t_1)} \oplus \Phi^{-1}(\nu, \delta^0\mathbf{0}) \cdot \chi_{[t_1,t_2)} \oplus \Phi^{-1}(\nu, \delta^0\delta^1) \cdot \chi_{[t_2,t_3)} \oplus ...$$

$$... \oplus \Phi^{-1}(\nu, \delta^0...\delta^{k-1}\mathbf{0}) \cdot \chi_{[t_{2k-1}, t_{2k})} \oplus \Phi^{-1}(\nu, \delta^0...\delta^k) \cdot \chi_{[t_{2k},t_{2k+1})} \oplus ...$$

We are interested in the following to characterize f^{-1} as 'inverse' of f. The system $f^{-1} \circ f$, see Figure 9, is obtained by putting $z_1 = x_1, ..., z_4 = x_4$ in equations (28),...,(32). In order to make a clear distinction between f and f^{-1}, we have indicated the sense of rotation of the rotating bar, counterclockwise for f and clockwise for f^{-1}. In the following Definition we make use of the fact that $\forall \mu \in \Theta_0, \forall u \in U, f_\mu(u) \in S \times U$, Theorem 31.

Definition 41 *We define the system $f^{-1} \circ f : U \to P^*(S^{(6)})$ by $\forall u \in U$,*

$$(f^{-1} \circ f)(u) = \{(x_0, v_0, v_1, v_2, v_3, v_4) | x \in f(u), v \in f^{-1}(x_1, x_2, x_3, x_4)\}.$$

Theorem 42 *For any $\mu, \nu \in \Theta_0$, we suppose that $z_1 = x_1, ..., z_4 = x_4$ and that at the time instant $t \in \mathbf{R}$, f_μ and f_ν^{-1} are in equilibrium:*

$$u_1(t-0) = ... = u_4(t-0) = x_1(t-0) = ... \tag{43}$$

$$... = x_4(t-0) = v_1(t-0) = ... = v_4(t-0) = 0.$$

a) If $\alpha \in \mathbf{B}$ exists such that

$$x_0(t-0) = v_0(t-0) = \alpha \tag{44}$$

and if t is a point of continuity of x_0,

$$x_0(t) = \alpha, \tag{45}$$

then t is a point of continuity of v_0 also,

$$v_0(t) = \alpha \tag{46}$$

and

$$v_1(t) = u_1(t), ..., v_4(t) = u_4(t). \tag{47}$$

b) If $\alpha \in \mathbf{B}$ exists such that

$$x_0(t - 0) = \alpha, \tag{48}$$

$$v_0(t - 0) = \overline{\alpha} \tag{49}$$

hold and if t is a point of discontinuity of x_0,

$$x_0(t) = \overline{\alpha}, \tag{50}$$

then t is a point of discontinuity of v_0 also,

$$v_0(t) = \alpha \tag{51}$$

and

$$v_1(t) = u_1(t), ..., v_4(t) = u_4(t). \tag{52}$$

is true.

c) If $\alpha \in \mathbf{B}$ exists such that

$$x_0(t - 0) = v_0(t - 0) = \alpha \tag{53}$$

holds and if t is a point of discontinuity for x_0,

$$x_0(t) = \overline{\alpha}, \tag{54}$$

then t is a point of continuity for v_0

$$v_0(t) = \alpha \tag{55}$$

and

$$v_1(t) = u_2(t), v_2(t) = u_3(t), v_3(t) = u_4(t), v_4(t) = u_1(t). \tag{56}$$

is true.

d) If $\alpha \in \mathbf{B}$ exists such that

$$x_0(t - 0) = \alpha, \tag{57}$$

$$v_0(t-0)=\overline{\alpha} \tag{58}$$

and if t *is a point of continuity for* x_0,

$$x_0(t)=\alpha, \tag{59}$$

then either t *is a point of continuity for* v_0 *also*

$$v_0(t)=\overline{\alpha}, \tag{60}$$

or t *is a point of discontinuity for* v_0

$$v_0(t)=\alpha; \tag{61}$$

in both cases we have

$$v_1(t)=u_4(t), v_2(t)=u_1(t), v_3(t)=u_2(t), v_4(t)=u_3(t). \tag{62}$$

Remark 43 *Like at Definition 41, we can define the system* $f\circ f^{-1}: U\to P^*(S^{(6)})$ *by* $\forall z\in U$,

$$(f\circ f^{-1})(z)=\{(v_0,x_0,x_1,x_2,x_3,x_4)|v\in f^{-1}(z), x\in f(v_1,v_2,v_3,v_4)\}.$$

We can also state a theorem similar with Theorem 42, where $z_1=x_1,...,z_4=x_4$ *are replaced by* $u_1=v_1,...,u_4=v_4$ *conformly with the replacement of* $f^{-1}\circ f$ *with* $f\circ f^{-1}$.

The conclusion resulting from the previous theorems is that the properties $\forall u\in U$, $\forall(x_0,v_0,v_1,v_2,v_3,v_4)\in(f^{-1}\circ f)(u)$,

$$u_1=v_1, u_2=v_2, u_3=v_3, u_4=v_4$$

and $\forall z\in U$, $\forall(v_0,x_0,x_1,x_2,x_3,x_4)\in(f\circ f^{-1})(z)$,

$$z_1=x_1, z_2=x_2, z_3=x_3, z_4=x_4$$

are not true, the inverse behavior of f *and* f^{-1} *is restricted to the cases a) and b) only of Theorem 42.*

9 The time invariance and the non-anticipation properties

Theorem 44 *The systems* f *and* f^{-1} *are time invariant, i.e. for any* $u,z\in U$ *and* $d\in\mathbf{R}$ *we have*

$$f(u\circ\tau^d)=\{x\circ\tau^d|x\in f(u)\},$$

$$f^{-1}(z\circ\tau^d)=\{v\circ\tau^d|v\in f^{-1}(z)\}.$$

Theorem 45 *The system f is non-anticipatory in the sense that for all $u \in U$ and all $x \in f(u)$ it satisfies one of the following statements:*

a) x is constant;

b) u, x are both variable and we have

$$\min\{t|u(t-0) \neq u(t)\} \leq \min\{t|x(t-0) \neq x(t)\}. \tag{63}$$

The same non-anticipation property is fulfilled by f^{-1} too.

Theorem 46 *f fulfills the non-anticipation properties: $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,*

$$u_{|(-\infty,t]} = v_{|(-\infty,t]} \Longrightarrow \{x_{|(-\infty,t]}|x \in f(u)\} = \{y_{|(-\infty,t]}|y \in f(v)\}; \tag{64}$$

$\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u_{|(-\infty,t]} = v_{|(-\infty,t]} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}. \tag{65}$$

The system f^{-1} fulfills these non-anticipation properties too.

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