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EVENTUALLY PERIODIC POINTS OF THE BINARY SIGNALS: DEFINITION, ACCESSIBILITY AND LIMIT OF PERIODICITY

Şerban E. Vlad
Oradea City Hall, Oradea, Romania
serban_e_vlad@yahoo.com

Abstract The discrete time and the real time binary signals model the electrical signals from the digital electrical engineering. Our present purpose is that of studying the eventually periodic points of the binary signals, i.e. the points $\mu \in \{0, 1\}^n$ whose periodicity starts at a time instant called limit of periodicity that is $\geq$ the initial time instant. The paper has three results. First, it gives several properties that are equivalent with the eventual periodicity of a point μ; second, it shows that an eventually periodic point is accessed at least once in a time interval with the length of a period; third, it characterizes the set of the limits of periodicity.

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1. INTRODUCTION

The electrical signals from the digital electrical engineering are modelled by discrete time and real time binary functions, called (binary) signals too. Our present purpose is to introduce the study of the eventual periodicity of the points of the signals by using a bibliography that consists in general in monographs on (real, usual) dynamical systems, treating occasionally periodicity. We have used analogies suggested by [1], [2], [4] concerning the periodic points and by [2], [3], [4] concerning the eventually periodic points, with the remarks:

- we have systems theory in such works and no systems theory here;
- we have real numbers in the cited monographs and the binary numbers $0, 1$ here;
- using the binary numbers is not essential in this context, the binary signals may be considered from the periodicity point of view as functions that take a finite number of values.

We give several equivalent properties that define the eventual periodicity of a point, we show that each eventually periodic point is accessed at least once during an arbitrary interval with the length of a period and finally we characterize, in certain circumstances, the set of the limits of periodicity = time instants wherefrom the periodicity of a point exists.

2. PRELIMINARIES

Notation 2.1. *The binary Boole algebra is denoted by* $\mathbf{B} = \{0, 1\}$. *Its laws are the usual ones: the logical complement '—', the product '* $\cdot$ *', the union'* $\cup$*', the modulo 2 sum '*$\oplus$*' and they induce laws that are denoted with the same symbols on* $\mathbf{B}^n, n \geq 1$.

Definition 2.1. *The sets* $\mathbf{B}$ *and* $\mathbf{B}^n$ *are organized as topological spaces by the discrete topology.*

Notation 2.2. *We use the notation* $\mathbf{N}_{\_} = \{-1, 0, 1, ...\}$ *for the discrete time set.*

Notation 2.3. *We denote*

$$\widehat{Seq} = \{(k_j) | k_j \in \mathbf{N}_{\_}, j \in \mathbf{N}_{\_} \text{ and } k_{-1} < k_0 < k_1 < ...\},$$

$$Seq = \{(t_k) | t_k \in \mathbf{R}, k \in \mathbf{N} \text{ and } t_0 < t_1 < t_2 < ... \text{ superiorly unbounded}\}.$$

Notation 2.4. *The characteristic function of the set* $A \subset \mathbf{R}$ *is denoted by* $\chi_A : \mathbf{R} \to \mathbf{B}$ $: \forall t \in \mathbf{R}$,

$$\chi_A(t) = \begin{cases} 1, if \ t \in A, \\ 0, otherwise. \end{cases}$$

Definition 2.2. *The* ***discrete time signals*** *are by definition the functions* $\widehat{x} : \mathbf{N}_{\_} \to \mathbf{B}^n$. *Their set is denoted with* $\widehat{S}^{(n)}$.

The ***continuous time signals*** *are the functions* $x : \mathbf{R} \to \mathbf{B}^n$ *of the form* $\forall t \in \mathbf{R}$,

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... \tag{1}$$

where $\mu \in \mathbf{B}^n$ *and* $(t_k) \in Seq$. *Their set is denoted by* $S^{(n)}$.

Remark 2.1. *The signals model the electrical signals of the circuits from the digital electrical engineering.*

Remark 2.2. *Throughout the paper the hat '*$\widehat{\ }$*' (that we have already used several times) indicates the discrete time.*

Remark 2.3. *The discrete time signals are sequences. The real time signals are piecewise constant functions.*

Lemma 2.1. *For any* $x \in S^{(n)}$ *and any* $t \in \mathbf{R}$, *we have the existence of* $x(t-0), x(t+0) \in \mathbf{B}^n$ *with the property*

$$\exists \varepsilon > 0, \forall \xi \in (t - \varepsilon, t), x(\xi) = x(t - 0), \tag{2}$$

$$\exists \varepsilon > 0, \forall \xi \in (t, t + \varepsilon), x(\xi) = x(t + 0). \tag{3}$$

Proof. We presume that x, t are arbitrary and fixed and that x fulfills (1) with $\mu \in \mathbf{B}^n$ and $(t_k) \in Seq$. We notice that if $t \leq t_0$, then $\varepsilon > 0$ arbitrary and $x(t - 0) = \mu$ fulfill

(2); and if $t \in (t_k, t_{k+1}], k \in \mathbf{N}$ then $\varepsilon \in (0, t - t_k)$ arbitrary and $x(t-0) = x(t_k)$ fulfill (2) too.

Similarly, if $t < t_0$, then any $\varepsilon \in (0, t_0 - t)$ and $x(t+0) = \mu$ fulfill (3); and if $t \in [t_k, t_{k+1}), k \in \mathbf{N}$ then $\varepsilon \in (0, t_{k+1} - t)$ arbitrary and $x(t+0) = x(t_k)$ fulfill (3) too. ∎

Definition 2.3. *The functions $\mathbf{R} \ni t \to x(t-0) \in \mathbf{B}^n$, $\mathbf{R} \ni t \to x(t+0) \in \mathbf{B}^n$ are called the **left limit** function of x and the **right limit** function of x.*

Remark 2.4. *Lemma 2.1 states that the left limit and the right limit functions of $x \in S^{(n)}$ exist. Moreover, from the proof of the Lemma we infer, see (1): $\forall t \in \mathbf{R}$,*

$$x(t-0) = \mu \cdot \chi_{(-\infty,t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0,t_1]}(t) \oplus ... \oplus x(t_k) \cdot \chi_{(t_k,t_{k+1}]}(t) \oplus ...$$

$$x(t) = x(t+0).$$

In particular we notice that $x(t-0)$ is not a signal.

Remark 2.5. *The existence of $x(t+0)$ is sometimes used under the form: $\forall t \in \mathbf{R}$,*

$$\exists \varepsilon > 0, \forall \xi \in [t, t+\varepsilon), x(\xi) = x(t).$$

Definition 2.4. *The **discrete time forgetful function** $\widehat{\sigma}^{k'} : \widehat{S}^{(n)} \to \widehat{S}^{(n)}$ is defined for $k' \in \mathbf{N}$ by*

$$\forall \widehat{x} \in \widehat{S}^{(n)}, \forall k \in \mathbf{N}_{\_}, \widehat{\sigma}^{k'}(\widehat{x})(k) = \widehat{x}(k+k')$$

*and the **real time forgetful function** $\sigma^{t'} : S^{(n)} \to S^{(n)}$ is defined for $t' \in \mathbf{R}$ in the following manner*

$$\forall x \in S^{(n)}, \forall t \in \mathbf{R}, \sigma^{t'}(x)(t) = \begin{cases} x(t), t \geq t', \\ x(t'-0), t < t'. \end{cases}$$

Remark 2.6. *Let us give $\widehat{x} \in \widehat{S}^{(n)}$ by its values $\widehat{x} = \widehat{x}(-1), \widehat{x}(0), \widehat{x}(1), ...$ Then $\widehat{\sigma}^{k'}(\widehat{x}) = \widehat{x}(k'-1), \widehat{x}(k'), \widehat{x}(k'+1), ...$ i.e. $\widehat{x}$ has forgotten its first k' values. In particular, $\widehat{x}$ forgets no value for $k' = 0$.*

Similarly, $\sigma^{t'}(x)$ makes x forget its values prior to t'; in particular, no value is forgotten if $\forall t < t', x(t) = \mu$.

Definition 2.5. *The **orbits** of $\widehat{x} \in \widehat{S}^{(n)}, x \in S^{(n)}$ are the sets of the values of these functions:*

$$\widehat{Or}(\widehat{x}) = \{\widehat{x}(k) | k \in \mathbf{N}_{\_}\},$$

$$Or(x) = \{x(t) | t \in \mathbf{R}\}.$$

Definition 2.6. *The **omega limit set** $\widehat{\omega}(\widehat{x})$ of $\widehat{x}$ is defined as*

$$\widehat{\omega}(\widehat{x}) = \{\mu | \mu \in \mathbf{B}^n, \exists (k_j) \in \widehat{Seq}, \forall j \in \mathbf{N}_{\_}, \widehat{x}(k_j) = \mu\}$$

*and the **omega limit set** $\omega(x)$ of x is defined by*

$$\omega(x) = \{\mu | \mu \in \mathbf{B}^n, \exists (t_k) \in Seq, \forall k \in \mathbf{N}, x(t_k) = \mu\}.$$

*The points of $\widehat{\omega}(\widehat{x}), \omega(x)$ are called **omega limit points**.*

Definition 2.7. *For $\widehat{x} \in \widehat{S}^{(n)}, x \in S^{(n)}$ and $\mu \in \mathbf{B}^n$, we define the **support sets** of μ by*

$$\widehat{\mathbf{T}}_\mu^{\widehat{x}} = \{k | k \in \mathbf{N}_\_, \widehat{x}(k) = \mu\},$$

$$\mathbf{T}_\mu^x = \{t | t \in \mathbf{R}, x(t) = \mu\}.$$

Definition 2.8. *By definition, the **initial time** instant of $\widehat{x} \in \widehat{S}^{(n)}$ is $k' = -1$. A point $t' \in \mathbf{R}$ is called **initial time** instant of $x \in S^{(n)}$ if*

$$\forall t \leq t', x(t) = x(t').$$

The set of the initial time instants of x is denoted by I^x.

Definition 2.9. *The **initial value** of $\widehat{x} \in \widehat{S}^{(n)}$ is by definition $\widehat{x}(-1) \in \mathbf{B}^n$. We denote by $x(-\infty + 0) \in \mathbf{B}^n$ the **initial value** of $x \in S^{(n)}$ which is defined this way:*

$$\forall t \in I^x, x(t) = x(-\infty + 0).$$

Definition 2.10. *Let the signals $\widehat{x} \in \widehat{S}^{(n)}$ and $x \in S^{(n)}$. We say that $\mu \in \widehat{\omega}(\widehat{x})$ is **periodic** with the **period** $p \geq 1$ if*

$$\forall k \in \widehat{\mathbf{T}}_\mu^{\widehat{x}}, \{k + zp | z \in \mathbf{Z}\} \cap \mathbf{N}_\_ \subset \widehat{\mathbf{T}}_\mu^{\widehat{x}} \tag{4}$$

*and we also say that $\mu \in \omega(x)$ is **periodic** with the **period** $T > 0$ if $t' \in I^x$ exists such that*

$$\forall t \in \mathbf{T}_\mu^x \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}_\mu^x. \tag{5}$$

Remark 2.7. *The general form (1) of x shows that $I^x \neq \emptyset$ and $I^x = \mathbf{R} \Longleftrightarrow x$ is constant.*

Remark 2.8. *The finiteness of $\widehat{Or}(\widehat{x}), Or(x)$ implies the existence of the omega limit points: $\emptyset \neq \widehat{\omega}(\widehat{x}) \subset \widehat{Or}(\widehat{x}), \emptyset \neq \omega(x) \subset Or(x)$.*

Remark 2.9. *If we ask that the point $\mu \in \widehat{Or}(\widehat{x}), \mu \in Or(x)$ is periodic, in the sense that (4), (5) are fulfilled, then $\mu \in \widehat{\omega}(\widehat{x}), \mu \in \omega(x)$. The supposition from the very beginning that the periodic point μ is an omega limit point that fulfills a certain property does not restrict the generality. It is convenient to state the definition of periodicity for omega limit points only because this way we avoid asking further requests of nontriviality like $\widehat{\mathbf{T}}_\mu^{\widehat{x}} \cap \{k', k'+1, k'+2, ...\} \neq \emptyset, k' \in \mathbf{N}_\_$ and $\mathbf{T}_\mu^x \cap [t', \infty) \neq \emptyset$, $t' \in \mathbf{R}$, which are obviously satisfied by $\mu \in \widehat{\omega}(\widehat{x}), \mu \in \omega(x)$.*

Remark 2.10. *We interpret (4) this way. The periodicity of μ with the period p means that for any k with $\widehat{x}(k) = \mu$, if we go upwards $k + p, k + 2p, ...$ or downwards $k - p, k - 2p, ...$ with multiples of p, without getting out of the discrete time set $\mathbf{N}_\_$, we get a time instant where $\widehat{x}$ equals μ.*

And we interpret (5) similarly, by replacing the initial time -1 from (4) with the initial time $t' \in I^x$. The periodicity of μ with the period T means that for any $t \geq t'$ with $x(t) = \mu$, if we go upwards $t + T, t + 2T, ...$ or downwards $t - T, t - 2T, ...$ with multiples of T without getting out of the set $[t', \infty)$, we get a time instant where x equals μ.

3. EVENTUALLY PERIODIC POINTS

Theorem 3.1. *a) Let $\widehat{x} \in \widehat{S}^{(n)}$, $\mu \in \widehat{\omega}(\widehat{x})$, $p \geq 1$ and $k' \in \mathbf{N}_\_$. The following statements are equivalent*:

$$\begin{cases} \forall k \in \widehat{\mathbf{T}}^{\widehat{x}}_\mu \cap \{k', k'+1, k'+2, ...\}, \\ \{k + zp | z \in \mathbf{Z}\} \cap \{k', k'+1, k'+2, ...\} \subset \widehat{\mathbf{T}}^{\widehat{x}}_\mu, \end{cases} \tag{6}$$

$$\forall k \in \widehat{\mathbf{T}}^{\widehat{\sigma}^{k'+1}(\widehat{x})}_\mu, \{k + zp | z \in \mathbf{Z}\} \cap \mathbf{N}_\_ \subset \widehat{\mathbf{T}}^{\widehat{\sigma}^{k'+1}(\widehat{x})}_\mu, \tag{7}$$

$$\begin{cases} \forall k \geq k', \widehat{x}(k) = \mu \Longrightarrow \\ \Longrightarrow (\widehat{x}(k) = \widehat{x}(k+p) \text{ and } k - p \geq k' \Longrightarrow \widehat{x}(k) = \widehat{x}(k-p)), \end{cases} \tag{8}$$

$$\begin{cases} \forall k \in \mathbf{N}_\_, \widehat{\sigma}^{k'+1}(\widehat{x})(k) = \mu \Longrightarrow \\ \Longrightarrow (\widehat{\sigma}^{k'+1}(\widehat{x})(k) = \widehat{\sigma}^{k'+1}(\widehat{x})(k+p) \text{ and} \\ k - p \geq -1 \Longrightarrow \widehat{\sigma}^{k'+1}(\widehat{x})(k) = \widehat{\sigma}^{k'+1}(\widehat{x})(k-p)). \end{cases} \tag{9}$$

b) Let $x \in S^{(n)}$, $\mu \in \omega(x)$, $T > 0$ and $t' \in \mathbf{R}$. We have the equivalence of the following statements:

$$\forall t \in \mathbf{T}^x_\mu \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}^x_\mu, \tag{10}$$

$$\begin{cases} \exists \varepsilon > 0, \forall t'' \in (t', t' + \varepsilon), t' \in I^{\sigma^{t''}(x)} \text{ and} \\ \forall t \in \mathbf{T}^{\sigma^{t''}(x)}_\mu \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}^{\sigma^{t''}(x)}_\mu, \end{cases} \tag{11}$$

$$\begin{cases} \forall t \geq t', x(t) = \mu \Longrightarrow \\ \Longrightarrow (x(t) = x(t+T) \text{ and } t - T \geq t' \Longrightarrow x(t) = x(t-T)), \end{cases} \tag{12}$$

$$\begin{cases} \exists \varepsilon > 0, \forall t'' \in (t', t' + \varepsilon), t' \in I^{\sigma^{t''}(x)} \text{ and} \\ \forall t \geq t', \sigma^{t''}(x)(t) = \mu \Longrightarrow (\sigma^{t''}(x)(t) = \sigma^{t''}(x)(t+T) \text{ and} \\ t - T \geq t' \Longrightarrow \sigma^{t''}(x)(t) = \sigma^{t''}(x)(t-T)). \end{cases} \tag{13}$$

Proof. a) The following formula

$$\widehat{\mathbf{T}}^{\widehat{\sigma}^{k'+1}(\widehat{x})}_\mu = \{k | k \in \mathbf{N}_\_, \widehat{x}(k + k' + 1) = \mu\} = \widehat{\mathbf{T}}^{\widehat{x}}_\mu \cap \{k', k'+1, k'+2, ...\}$$

holds and the supposition that $\mu \in \widehat{\omega}(\widehat{x})$ implies, since $\widehat{\mathbf{T}_\mu^x}$ is infinite, that all the previous sets are non-empty.

(6)$\Longrightarrow$(7) Let $k \in \widehat{\mathbf{T}}_\mu^{\widehat{\sigma}^{k'+1}(\widehat{x})}$ and $z \in \mathbf{Z}$ arbitrary with $k + zp \geq -1$. We have $\widehat{\sigma}^{k'+1}(\widehat{x})(k) = \widehat{x}(k + k' + 1) = \mu$, $k + k' + 1 \geq k'$ and $k + k' + 1 + zp \geq k'$, thus $k + k' + 1 + zp \overset{(6)}{\in} \widehat{\mathbf{T}_\mu^x}$, wherefrom $\widehat{\sigma}^{k'+1}(\widehat{x})(k + zp) = \widehat{x}(k + k' + 1 + zp) = \mu$ and the conclusion is $k + zp \in \widehat{\mathbf{T}}_\mu^{\widehat{\sigma}^{k'+1}(\widehat{x})}$.

(7)$\Longrightarrow$(8) Let $k \geq k'$ arbitrary with $\widehat{x}(k) = \mu$. We can see that $k - k' - 1 \geq -1$, $\mu = \widehat{\sigma}^{k'+1}(\widehat{x})(k - k' - 1)$ and

$$k - k' - 1 + p \in \{k - k' - 1 + zp | z \in \mathbf{Z}\} \cap \mathbf{N}_{\_} \overset{(7)}{\subset} \widehat{\mathbf{T}}_\mu^{\widehat{\sigma}^{k'+1}(\widehat{x})}$$

hold. It has resulted that $\mu = \widehat{\sigma}^{k'+1}(\widehat{x})(k - k' - 1 + p) = \widehat{x}(k + p)$.

If, in addition, $k - p \geq k'$, then $k - k' - 1 - p \geq -1$ and

$$k - k' - 1 - p \in \{k - k' - 1 + zp | z \in \mathbf{Z}\} \cap \mathbf{N}_{\_} \overset{(7)}{\subset} \widehat{\mathbf{T}}_\mu^{\widehat{\sigma}^{k'+1}(\widehat{x})},$$

in other words $\mu = \widehat{\sigma}^{k'+1}(\widehat{x})(k - k' - 1 - p) = \widehat{x}(k - p)$.

(8)$\Longrightarrow$(9) Let $k \in \widehat{\mathbf{T}}_\mu^{\widehat{\sigma}^{k'+1}(\widehat{x})}$ arbitrary. We have $k \geq -1$, $k + k' + 1 \geq k'$ and we can apply (8):

$$\widehat{\sigma}^{k'+1}(\widehat{x})(k) = \widehat{x}(k + k' + 1) \overset{(8)}{=} \widehat{x}(k + k' + 1 + p) = \widehat{\sigma}^{k'+1}(\widehat{x})(k + p).$$

If in addition $k - p \geq -1$, then $k - p + k' + 1 \geq k'$ and we can apply (8) again:

$$\widehat{\sigma}^{k'+1}(\widehat{x})(k) = \widehat{x}(k + k' + 1) \overset{(8)}{=} \widehat{x}(k + k' + 1 - p) = \widehat{\sigma}^{k'+1}(\widehat{x})(k - p).$$

(9)$\Longrightarrow$(6) Let $k \in \widehat{\mathbf{T}_\mu^x}$, $z \in \mathbf{Z}$ arbitrary, having the property that $k \geq k', k + zp \geq k'$. We have $k - k' - 1 \geq -1, k + zp - k' - 1 \geq -1$ and the following possibilities result.

Case $z > 0$,

$$\mu = \widehat{x}(k) = \widehat{\sigma}^{k'+1}(\widehat{x})(k - k' - 1) \overset{(9)}{=} \widehat{\sigma}^{k'+1}(\widehat{x})(k - k' - 1 + p) \overset{(9)}{=} ...$$

$$... \overset{(9)}{=} \widehat{\sigma}^{k'+1}(\widehat{x})(k - k' - 1 + (z - 1)p) \overset{(9)}{=} \widehat{\sigma}^{k'+1}(\widehat{x})(k - k' - 1 + zp) = \widehat{x}(k + zp);$$

Case $z = 0$,

$$\mu = \widehat{x}(k) = \widehat{x}(k + zp);$$

Case $z < 0$,

$$\mu = \widehat{x}(k) = \widehat{\sigma}^{k'+1}(\widehat{x})(k - k' - 1) \overset{(9)}{=} \widehat{\sigma}^{k'+1}(\widehat{x})(k - k' - 1 - p) \overset{(9)}{=} ...$$

$$... \overset{(9)}{=} \widehat{\sigma}^{k'+1}(\widehat{x})(k - k' - 1 + (z + 1)p) \overset{(9)}{=} \widehat{\sigma}^{k'+1}(\widehat{x})(k - k' - 1 + zp) = \widehat{x}(k + zp).$$

We have obtained in all the three cases that $k + zp \in \widehat{\mathbf{T}_\mu^x}$.

b) Let $\varepsilon > 0$ with the property

$$\forall \xi \in [t', t' + \varepsilon), x(\xi) = x(t') \tag{14}$$

and we take $t'' \in (t', t' + \varepsilon)$ arbitrarily. For $\varepsilon' \in (0, t'' - t')$ we see that $\forall \xi \in (t'' - \varepsilon', t''), x(\xi) = x(t')$, thus $x(t'' - 0) = x(t')$ and we have

$$\sigma^{t''}(x)(t) = \begin{cases} x(t), t \geq t'' \\ x(t'' - 0), t < t'' \end{cases} = \begin{cases} x(t), t \geq t' \\ x(t'), t < t' + \varepsilon \end{cases}. \tag{15}$$

We notice that $t' \in I^{\sigma^{t''}(x)}$, since $\forall t \leq t', \sigma^{t''}(x)(t) = x(t') = \sigma^{t''}(x)(t')$ and the following formula

$$\mathbf{T}_\mu^{\sigma^{t''}(x)} \cap [t', \infty) = \{t | t \geq t', \sigma^{t''}(x)(t) = \mu\} = \mathbf{T}_\mu^x \cap [t', \infty)$$

holds. The request $\mu \in \omega(x)$ implies that all the previous sets are non-empty, since $\mathbf{T}_\mu^x$ is superiorly unbounded.

(10)$\Longrightarrow$(11) We take $t \in \mathbf{T}_\mu^{\sigma^{t''}(x)}$ and $z \in \mathbf{Z}$ arbitrary with the property that $t \geq t'$ and $t + zT \geq t'$. We have

$$\mu = \sigma^{t''}(x)(t) \overset{(15)}{=} x(t) \overset{(10)}{=} x(t + zT) \overset{(15)}{=} \sigma^{t''}(x)(t + zT).$$

(11)$\Longrightarrow$(12) Let $t \geq t'$ arbitrary with the property $x(t) = \mu$. As

$$\sigma^{t''}(x)(t) \overset{(15)}{=} x(t) = \mu,$$

we can apply (11), thus

$$t + T \in \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \overset{(11)}{\subset} \mathbf{T}_\mu^{\sigma^{t''}(x)}$$

and

$$\mu = \sigma^{t''}(x)(t + T) \overset{(15)}{=} x(t + T).$$

If in addition $t - T \geq t'$, then

$$t - T \in \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \overset{(11)}{\subset} \mathbf{T}_\mu^{\sigma^{t''}(x)}$$

and

$$\mu = \sigma^{t''}(x)(t - T) \overset{(15)}{=} x(t - T).$$

(12)$\Longrightarrow$(13) We have for $t \geq t'$ arbitrary with $\mu = \sigma^{t''}(x)(t) = x(t)$ that

$$\mu = \sigma^{t''}(x)(t) = x(t) \overset{(12)}{=} x(t + T) = \sigma^{t''}(x)(t + T)$$

etc.

(13)$\Longrightarrow$(10) Let $t \in \mathbf{T}^x_\mu$ and $z \in \mathbf{Z}$ with the property that $t \geq t'$ and $t + zT \geq t'$. We have $\sigma^{t''}(x)(t) = x(t) = \mu$, thus we can apply (13).

Case $z > 0$,

$$\mu = \sigma^{t''}(x)(t) \overset{(13)}{=} \sigma^{t''}(x)(t+T) \overset{(13)}{=} ...$$

$$... \overset{(13)}{=} \sigma^{t''}(x)(t+(z-1)T) \overset{(13)}{=} \sigma^{t''}(x)(t+zT) = x(t+zT);$$

Case $z = 0$,

$$\mu = x(t) = x(t+zT);$$

Case $z < 0$,

$$\mu = \sigma^{t''}(x)(t) \overset{(13)}{=} \sigma^{t''}(x)(t-T) \overset{(13)}{=} ...$$

$$... \overset{(13)}{=} \sigma^{t''}(x)(t+(z+1)T) \overset{(13)}{=} \sigma^{t''}(x)(t+zT) = x(t+zT).$$

In all the three cases we have inferred $t + zT \in \mathbf{T}^x_\mu$. ∎

Definition 3.1. *We consider the signal $\widehat{x}$ and the point $\mu \in \widehat{\omega}(\widehat{x})$. If $p \geq 1$ and $k' \in \mathbf{N}_\_$ exist such that one of (6),..., (9) holds, then μ is said to be **eventually periodic** (an **eventually periodic point of** $\widehat{x}$, or **of** $\widehat{Or}(\widehat{x})$) with the **period** p and with the **limit of periodicity** k'. The least p, k' are called **prime period** and **prime limit of periodicity**.*

*Let x and $\mu \in \omega(x)$ and we suppose that $T > 0$ and $t' \in \mathbf{R}$ exist such that one of (10),..., (13) is true. Then μ is said to be **eventually periodic** (an **eventually periodic point of** x, or **of** $Or(x)$) with the **period** T and with the **limit of periodicity** t'. The least T, t' are called **prime period** and **prime limit of periodicity**.*

Remark 3.1. *Theorem 3.1 states that the eventual periodicity of $\mu \in \widehat{\omega}(\widehat{x}), \mu \in \omega(x)$ with the period p, T and the limit of periodicity k', t' coincides with the periodicity of $\mu \in \widehat{\omega}(\widehat{\sigma}^{k'+1}(\widehat{x})), \mu \in \omega(\sigma^{t''}(x))$ with the period p, T.*

Remark 3.2. *In the real time case, the prime period T and the prime limit of periodicity might not exist. This happens for example when x is constant.*

4. THE ACCESSIBILITY OF THE EVENTUALLY PERIODIC POINTS

Theorem 4.1. *a) Let $\widehat{x}$ and $\mu \in \widehat{\omega}(\widehat{x})$ that is eventually periodic, with the period $p \geq 1$ and the limit of periodicity $k' \in \mathbf{N}_\_$. For any $k \geq k'$ we have $\widehat{\mathbf{T}^x_\mu} \cap \{k, k+1, ..., k+p-1\} \neq \emptyset$.*

b) Let x and $\mu \in \omega(x)$ that is eventually periodic with the period $T > 0$ and the limit of periodicity $t' \in \mathbf{R}$. For any $t \geq t'$, we have $\mathbf{T}^x_\mu \cap [t, t+T) \neq \emptyset$.

Proof. a) The hypothesis implies the truth of

$$\widehat{\mathbf{T}^x_\mu} \cap \{k', k'+1, k'+2, ...\} \neq \emptyset, \tag{16}$$

$$\forall k \in \widehat{\mathbf{T}_{\mu}^{x}} \cap \{k', k'+1, k'+2, ...\},$$
$$\{k+zp|z \in \mathbf{Z}\} \cap \{k', k'+1, k'+2, ...\} \subset \widehat{\mathbf{T}_{\mu}^{x}}. \tag{17}$$

The statement (16) allows us to define $k'' = \min \widehat{\mathbf{T}_{\mu}^{x}} \cap \{k', k'+1, k'+2, ...\}$ and we prove that $k'' \in \widehat{\mathbf{T}_{\mu}^{x}} \cap \{k', k'+1, ..., k'+p-1\}$. If, against all reason, this would not be true, then we would have $k'' \geq k'+p$ and

$$k''-p \in \{k''+zp|z \in \mathbf{Z}\} \cap \{k', k'+1, k'+2, ...\} \overset{(17)}{\subset} \widehat{\mathbf{T}_{\mu}^{x}},$$

representing a contradiction with the definition of k''.

From (17) we infer that $\{k'', k''+p, k''+2p, ...\} \subset \widehat{\mathbf{T}_{\mu}^{x}} \cap \{k', k'+1, k'+2, ...\}$, meaning that $\forall k \geq k'$, $\widehat{\mathbf{T}_{\mu}^{x}} \cap \{k, k+1, ..., k+p-1\} \neq \emptyset$.

b) We have from the hypothesis that

$$\mathbf{T}_{\mu}^{x} \cap [t', \infty) \neq \emptyset, \tag{18}$$

$$\forall t \in \mathbf{T}_{\mu}^{x} \cap [t', \infty), \{t+zT|z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}_{\mu}^{x} \tag{19}$$

are fulfilled. From (1) and (18) we get that $\mathbf{T}_{\mu}^{x} \cap [t', \infty)$ has one of the forms $[a, b)$, $[a, \infty)$, $[a_1, b_1) \cup ... \cup [a_k, b_k)$, $[a_1, b_1) \cup ... \cup [a_k, b_k) \cup ...$, $[a_1, b_1) \cup ... \cup [a_k, b_k) \cup [a_{k+1}, \infty)$ and this allows defining $t'' = \min \mathbf{T}_{\mu}^{x} \cap [t', \infty)$. We show that $t'' \in \mathbf{T}_{\mu}^{x} \cap [t', t'+T)$. If, against all reason, this would not be true, then we would have $t'' \geq t'+T$. This means that $t''-T \geq t'$, thus

$$t''-T \in \{t''+zT|z \in \mathbf{Z}\} \cap [t', \infty) \overset{(19)}{\subset} \mathbf{T}_{\mu}^{x},$$

contradiction with the definition of t''.

By using (19) we get $\{t'', t''+T, t''+2T, ...\} \subset \mathbf{T}_{\mu}^{x} \cap [t', \infty)$. The statement of the Theorem holds. ∎

5. THE LIMIT OF PERIODICITY

Lemma 5.1. *a) $\widehat{x} \in \widehat{S}^{(n)}$ is given and we suppose that $\mu \in \widehat{\omega}(\widehat{x})$ is eventually periodic with the period $p \geq 1$ and with the limit of periodicity $k' \in \mathbf{N}_{\_}$. If $k'' \geq k'$, then μ is eventually periodic with the period p and with the limit of periodicity k''.*

b) Let $x \in S^{(n)}$ and we suppose that $\mu \in \omega(x)$ is eventually periodic with the period $T > 0$ and with the limit of periodicity $t' \in \mathbf{R}$. If $t'' \geq t'$, then μ is eventually periodic with the period T and with the limit of periodicity t''.

Proof. b) The hypothesis states that

$$\forall t \in \mathbf{T}_{\mu}^{x} \cap [t', \infty), \{t+zT|z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}_{\mu}^{x} \tag{20}$$

is true and we must prove

$$\forall t \in \mathbf{T}_{\mu}^{x} \cap [t'', \infty), \{t+zT|z \in \mathbf{Z}\} \cap [t'', \infty) \subset \mathbf{T}_{\mu}^{x} \tag{21}$$

for an arbitrary $t'' \geq t'$. Indeed, we take some arbitrary $t \in \mathbf{T}^x_\mu \cap [t'', \infty)$ and $z \in \mathbf{Z}$ such that $t + zT \geq t''$ holds. Then $t \in \mathbf{T}^x_\mu \cap [t', \infty)$ and $t + zT \geq t'$ are true, thus we can apply (20). We have obtained that $t + zT \in \mathbf{T}^x_\mu$, i.e. (21) is fulfilled. ∎

Theorem 5.1. *a)* $\widehat{x} \in \widehat{S}^{(n)}, \mu \in \widehat{\omega}(\widehat{x}), p \geq 1, p' \geq 1, k' \in \mathbf{N}_\_, k'' \in \mathbf{N}_\_$ *are given. If*

$$\begin{cases} \forall k \in \widehat{\mathbf{T}}^{\widehat{x}}_\mu \cap \{k', k'+1, k'+2, ...\}, \\ \{k + zp | z \in \mathbf{Z}\} \cap \{k', k'+1, k'+2, ...\} \subset \widehat{\mathbf{T}}^{\widehat{x}}_\mu, \end{cases} \tag{22}$$

$$\begin{cases} \forall k \in \widehat{\mathbf{T}}^{\widehat{x}}_\mu \cap \{k'', k''+1, k''+2, ...\}, \\ \{k + zp' | z \in \mathbf{Z}\} \cap \{k'', k''+1, k''+2, ...\} \subset \widehat{\mathbf{T}}^{\widehat{x}}_\mu \end{cases} \tag{23}$$

hold, then

$$\begin{cases} \forall k \in \widehat{\mathbf{T}}^{\widehat{x}}_\mu \cap \{k', k'+1, k'+2, ...\}, \\ \{k + zp' | z \in \mathbf{Z}\} \cap \{k', k'+1, k'+2, ...\} \subset \widehat{\mathbf{T}}^{\widehat{x}}_\mu \end{cases} \tag{24}$$

is true.

b) Let $x \in S^{(n)}, \mu \in \omega(x), T > 0, T' > 0, t' \in \mathbf{R}, t'' \in \mathbf{R}$. *Then*

$$\forall t \in \mathbf{T}^x_\mu \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}^x_\mu, \tag{25}$$

$$\forall t \in \mathbf{T}^x_\mu \cap [t'', \infty), \{t + zT' | z \in \mathbf{Z}\} \cap [t'', \infty) \subset \mathbf{T}^x_\mu \tag{26}$$

imply

$$\forall t \in \mathbf{T}^x_\mu \cap [t', \infty), \{t + zT' | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}^x_\mu. \tag{27}$$

Proof. b) Let $t \in \mathbf{T}^x_\mu, z \in \mathbf{Z}$ arbitrary such that $t \geq t'$ and $t + zT' \geq t'$. Such a t exists since $\mu \in \omega(x)$. We have the following possibilities.

Case $t' \geq t''$

Then $t \geq t''$ and $t + zT' \geq t''$, thus $t + zT' \overset{(26)}{\in} \mathbf{T}^x_\mu$ (see also Lemma 5.1).

Case $t' < t''$

$k \in \mathbf{N}$ exists with $t+kT \geq t'', t+zT'+kT \geq t''$. Obviously $t+kT \geq t', t+zT'+kT \geq t'$ and we can write

$$\mu = x(t) \overset{(25)}{=} x(t + kT) \overset{(26)}{=} x(t + zT' + kT) \overset{(25)}{=} x(t + zT'),$$

in other words $t + zT' \in \mathbf{T}^x_\mu$. ∎

Remark 5.1. *The previous Theorem states that the set of the limits of periodicity does not depend on the period.*

Notation 5.1. *We denote*

$$\widehat{L}^{\widehat{x}}_\mu = \{k' | k' \in \mathbf{N}_\_, \exists p \geq 1, (6) \text{ holds}\},$$

$$L^x_\mu = \{t' | t' \in \mathbf{R}, \exists T > 0, (10) \text{ holds}\}.$$

Example 5.1. *Let the signal $x \in S^{(1)}$,*

$$x(t) = \chi_{[0,1)}(t) \oplus \chi_{[4,5)}(t) \oplus \chi_{[6,7)}(t) \oplus \chi_{[8,9)}(t) \oplus \chi_{[10,11)}(t) \oplus ...$$

The point $1 \in \omega(x)$ is eventually periodic and $2, 4$ are two periods. We might be tempted to think that the sets of limits of periodicity are different for $T = 2$ and $T' = 4$, but this is not the case; in both situations (10) is fulfilled with $L^x_1 = [3, \infty)$.

Theorem 5.2. *a) Let $\widehat{x} \in \widehat{S}^{(n)}$ and the eventually periodic point $\mu \in \widehat{\omega}(\widehat{x})$. Then $k' \in \mathbf{N}_-$ exists with $\widehat{L^x_\mu} = \{k', k'+1, k'+2, ...\}$.*

b) Let $x \in S^{(n)}$ non constant and the eventually periodic point $\mu \in \omega(x)$. Then $t' \in \mathbf{R}$ exists such that $L^x_\mu = [t', \infty)$.

Proof. a) The hypothesis states $\widehat{L^x_\mu} \neq \emptyset$. The statement is a consequence of Lemma 5.1.

b) Because x is not constant, $t_0 \in \mathbf{R}$ exists with $I^x = (-\infty, t_0)$, i.e.

$$\forall t < t_0, x(t) = x(-\infty + 0), \tag{28}$$

$$x(t_0) \neq x(-\infty + 0). \tag{29}$$

We suppose that $L^x_\mu \neq \emptyset$ and that μ has the period $T > 0$.

b.i) We show first that $t_0 - T \notin L^x_\mu$ and we suppose against all reason that $t_0 - T \in L^x_\mu$. We have two possibilities.

Case $\mu = x(-\infty + 0)$

The hypothesis $t_0 - T \in L^x_\mu$ implies, as far as $t_0 - T \in \mathbf{T}^x_\mu$,

$$\mu = x(t_0 - T) = x(t_0),$$

representing a contradiction with (29).

Case $\mu \neq x(-\infty + 0)$

We infer from Theorem 4.1 that $\mathbf{T}^x_\mu \cap [t_0 - T, t_0) \neq \emptyset$, wherefrom $\mu = x(-\infty + 0)$, representing a contradiction.

b.ii) From b.i) and from Lemma 5.1, we draw the conclusion that L^x_μ has one of the forms $L^x_\mu = (t', \infty), L^x_\mu = [t', \infty)$, where $t' > t_0 - T$. We show that the first possibility cannot take place, thus we suppose against all reason that t' exists with $L^x_\mu = (t', \infty)$. We have the existence of $\varepsilon' > 0, \varepsilon'' > 0$ such that

$$\forall t \in (t', t' + \varepsilon'), x(t) = x(t'), \tag{30}$$

$$\forall t \in (t' + T, t' + T + \varepsilon''), x(t) = x(t' + T) \tag{31}$$

and let $\varepsilon \in (0, \min\{\varepsilon', \varepsilon''\})$. Two possibilities exist.

Case $x(t') = \mu$

We have $t' \notin L^x_\mu$, thus $x(t' + T) \neq \mu$ and $(t', t' + \varepsilon) \subset L^x_\mu$ means that

$$\forall t \in (t', t' + \varepsilon), x(t) = \mu \Longrightarrow x(t) = x(t + T). \tag{32}$$

Let $t \in (t', t' + \varepsilon)$ arbitrary. We can write

$$\mu = x(t') \overset{(30)}{=} x(t) \overset{(32)}{=} x(t+T) \overset{(31)}{=} x(t'+T),$$

contradiction.

Case $x(t') \neq \mu$

In this case two possibilities exist. The case $x(t'+T) = \mu$, when $(t', t'+\varepsilon) \subset L^x_\mu$ means the truth of (32). Let $t \in (t', t'+\varepsilon)$ arbitrary. We conclude

$$\mu = x(t'+T) \overset{(31)}{=} x(t+T) \overset{(32)}{=} x(t) \overset{(30)}{=} x(t'),$$

representing a contradiction. And the case $x(t'+T) \neq \mu$ when $\forall k \in \mathbf{N}, x(t+kT) \neq \mu$. As for any $t \in \mathbf{T}^x_\mu \cap (t', \infty) = \mathbf{T}^x_\mu \cap [t', \infty)$, we have $\{t+zT | z \in \mathbf{Z}\} \cap (t', \infty) = \{t+zT | z \in \mathbf{Z}\} \cap [t', \infty)$, the conclusion is $t' \in L^x_\mu$, contradiction.

It has resulted that the existence of $t' > t_0 - T$ with $L^x_\mu = [t', \infty)$ is the only possibility. ∎

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