ANALELE UNIVERSITATII DIN ORADEA
FASCICOLA MATEMATICA, TOM III, 1993

CURVILINEAR PSEUDOBOOLEAN INTEGRALS

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1. By $\boldsymbol{B}_2 = \{0,1,\oplus,\cdot\}$ we write the binary Boole algebra, where $'\oplus'$ is the modulo 2 sum and $'\cdot'$ is the intersection. If $\{a_i \mid i \in I\}$ is a binary family so that it has a finite support

$$|\{i \mid i \in I, a_i = 1\}| < \infty$$

then it makes sense to speak about the modulo 2 summation of the elements of this family, that will be written $\underset{i\in I}{\Xi}\, a_i$. In fact,

$$\underset{i\in I}{\Xi}\, a_i = \begin{cases} 1, if \ |\{i \mid i \in I, a_i = 1\}| = 2k+1 \\ 0, if \ |\{i \mid i \in I, a_i = 1\}| = 2k \end{cases}$$

If $I = \boldsymbol{N}$, then the meaning of this sum is that of a convergent series. If the support of this family to be summed is not finite, then the sum cannot be done and this is the sense of a divergent series in this theory.

The topology of $\boldsymbol{B}_2$ is the discrete one.

2. Let $x:[a,b] \to \boldsymbol{B}_2$ with the property that the set $\{t \mid t \in [a,b], x(t) = 1\}$ is finite. The *left integral* and the *right integral* of x on $[a,b)$, respectively on $(a,b]$ is given by:

$$\int_a^b x = \underset{\xi\in[a,b)}{\Xi} x(\xi), \int_a^{b\,*} x = \underset{\xi\in(a,b]}{\Xi} x(\xi)$$

3. A function $x:[a,b] \to \boldsymbol{B}_2$ has *left limits* if

$$\forall t \in (a,b], \exists x(t-0) \in \boldsymbol{B}_2, \exists \varepsilon > 0, \forall \xi \in (t-\varepsilon, t) \wedge (a,b], x(\xi) = x(t-0)$$

and, in a dual way, x has *right limits* if

$$\forall t \in [a,b), \exists x(t+0) \in \boldsymbol{B}_2, \exists \varepsilon > 0, \forall \xi \in (t, t+\varepsilon) \wedge [a,b), x(\xi) = x(t+0)$$

In the case that these properties are true, there are defined the functions $x(t-0)$ and $x(t+0)$ called the *left limit* and the *right limit function* of x.

4. Let $x:[a,b] \to \boldsymbol{B}_2$ be with left limits and right limits (not equal, in general). In such cases, we ask that in the points a and b, x has a right limit, respectively a left limit. The functions:

$$Dx(t) = x(t-0) \oplus x(t), D^*x(t) = x(t+0) \oplus x(t)$$

are called the *left derivative* and the *right derivative* of x.

5. **Lemma** If $x:[a,b] \to \boldsymbol{B}_2$ has left limits and right limits, then the sets

$$\{t \mid t \in (a,b], Dx(t) = 1\}, \{t \mid t \in [a,b), D^*x(t) = 1\}$$

are finite.

Proof From the definition of the left limits and of the right limits, we have the existence of a set of points

$$a = t_0 < t_1 < ... < t_n = b$$

with the property that

$$\forall i \in \{1,...,n\}, \forall \xi, \xi' \in (t_{i-1}, t_i), x(\xi) = x(\xi')$$

and finally

$$\{t \mid t \in (a,b], Dx(t) = 1\} \subset \{t_1,...,t_n\}, \{t \mid t \in [a,b), D^* x(t) = 1\} \subset \{t_0,...,t_{n-1}\}$$

6. **Remark** The previous definition 3 may be extended at $[a,b] \to \boldsymbol{B}_2^n$ functions, $n \geq 1$. In the paper, we shall consider the case $n = 2$.

7. A *path* in $\boldsymbol{B}_2^2$ is a function $\gamma : [a,b] \to \boldsymbol{B}_2^2$, $\gamma = (\gamma_1, \gamma_2)$, that has left limits and right limits.

The path is called *closed* if $\gamma(a) = \gamma(b)$.

The *inverse* of γ is defined to be the path $\gamma^- : [a,b] \to \boldsymbol{B}_2^2$,

$$\gamma^-(t) = \gamma(a + b - t)$$

The *reunion* of the paths $\gamma : [a,b] \to \boldsymbol{B}_2^2$, $\sigma : [b,c] \to \boldsymbol{B}_2^2$ that satisfy $\gamma(b) = \sigma(b)$ is defined by

$$\gamma \vee \sigma : [a,c] \to \boldsymbol{B}_2^2, (\gamma \vee \sigma)(t) = \begin{cases} \gamma(t), t \in [a,b] \\ \sigma(t), t \in [b,c] \end{cases}$$

The *equivalence* of the paths $\gamma : [a,b] \to \boldsymbol{B}_2^2$ and $\sigma : [c,d] \to \boldsymbol{B}_2^2$ is written $\gamma \sim \sigma$ and is given by the existence of an increasing homeomorphism $h : [a,b] \to [c,d]$ so that the next diagram is commutative:

$$\begin{tikzcd} {[a,b]} \arrow[rr, "h"] \arrow[dr, "\gamma"'] & & {[c,d]} \arrow[dl, "\sigma"] \\ & \boldsymbol{B}_2^2 & \end{tikzcd}$$

8. It is called *Boolean form* on $\boldsymbol{B}_2^2$ a function $\omega : \boldsymbol{B}_2^2 \to F(\boldsymbol{B}_2^2, \boldsymbol{B}_2)$ where we have noted

$$F(\boldsymbol{B}_2^2, \boldsymbol{B}_2) = \{f \mid f : \boldsymbol{B}_2^2 \to \boldsymbol{B}_2\}$$

so that for any $x, y \in \boldsymbol{B}_2^2, x = (x_1, x_2), y = (y_1, y_2), \omega$ is given by:

$$\omega(x)(y) = \omega_1(x) \cdot y_1 \oplus \omega_2(x) \cdot y_2 \oplus \omega_3(x) \cdot y_1 \cdot y_2$$

We just mention that with the notation

$$F(\boldsymbol{B}_2^n, \boldsymbol{B}_2) = \{f \mid f : \boldsymbol{B}_2^n \to \boldsymbol{B}_2\}$$

the Boolean forms on $\boldsymbol{B}_2^n$ are functions $\boldsymbol{B}_2^n \to F(\boldsymbol{B}_2^n, \boldsymbol{B}_2)$.

9. Let the Boolean form ω on $\boldsymbol{B}_2^2$ and $\gamma : [a,b] \to \boldsymbol{B}_2^2$ be a path. Because the sets

$$\{t \mid t \in (a,b], \omega_1(\gamma(t)) \cdot D\gamma_1(t) = 1\}, \{t \mid t \in [a,b), \omega_1(\gamma(t)) \cdot D^* \gamma_1(t) = 1\}$$

$$\{t \mid t \in (a,b], \omega_2(\gamma(t)) \cdot D\gamma_2(t) = 1\}, \{t \mid t \in [a,b), \omega_2(\gamma(t)) \cdot D^* \gamma_2(t) = 1\}$$

$$\{t \mid t \in (a,b], \omega_3(\gamma(t)) \cdot D\gamma_1(t) \cdot D\gamma_2(t) = 1\}, \{t \mid t \in [a,b), \omega_3(\gamma(t)) \cdot D^* \gamma_1(t) \cdot D^* \gamma_2(t) = 1\}$$

are finite – accordingly to lemma 5- there has sense the number

$$\int_\gamma \omega = \int_a^b \omega_1(\gamma)\cdot D^*\gamma_1 \oplus \int_a^b \omega_2(\gamma)\cdot D^*\gamma_2 \oplus \int_a^b \omega_3(\gamma)\cdot D^*\gamma_1 \cdot D^*\gamma_2 \oplus$$

$$\oplus \int_a^{b\,*} \omega_1(\gamma)\cdot D\gamma_1 \oplus \int_a^{b\,*} \omega_2(\gamma)\cdot D\gamma_2 \oplus \int_a^{b\,*} \omega_3(\gamma)\cdot D\gamma_1 \cdot D\gamma_2$$

all the six integrals from the right side being defined. It is called the *curvilinear integral of* ω *along the path* γ.

10. **Theorem** a) If $\gamma \sim \sigma$, then

$$\int_\gamma \omega = \int_\sigma \omega$$

b) If $\gamma \vee \sigma$ makes sense, then

$$\int_{\gamma\vee\sigma} \omega = \int_\gamma \omega \oplus \int_\sigma \omega$$

c)
$$\int_\gamma \omega = \int_{\gamma^-} \omega .$$

Proof a) Let $\gamma:[a,b]\to \boldsymbol{B}_2^2$ and $\sigma:[c,d]\to \boldsymbol{B}_2^2$ be two paths for which there exists the increasing homeomorphism $h:[a,b]\to[c,d]$ with the property that it makes commutative the diagram:

$$\begin{tikzcd}
{[a,b]} \arrow[rr, "h"] \arrow[dr, "\gamma"'] & & {[c,d]} \arrow[dl, "\sigma"] \\
& \boldsymbol{B}_2^2 &
\end{tikzcd}$$

The fact that there takes place

$$\int_a^{b\,*} \omega_1(\gamma)\cdot D\gamma_1 = \int_c^{d\,*} \omega_1(\sigma)\cdot D\sigma_1$$

for example may be shown by taking into account the following equivalences, where to $u,u',u''\in(a,b]$ there uniquely correspond the points $v=h(u), v'=h(u'), v''=h(u'')\in(c,d]$:

$$\omega_1(\gamma(u))\cdot D\gamma_1(u)=1 \Leftrightarrow \exists u'\in(a,u), \forall u''\in(u',u), \omega_1(\gamma(u))\cdot(\gamma_1(u'')\oplus\gamma_1(u))=1$$
$$\Leftrightarrow \exists v'\in(c,v), \forall v''\in(v',v), \omega_1(\sigma(v))\cdot(\sigma_1(v'')\oplus\sigma_1(v))=1 \Leftrightarrow \omega_1(\sigma(v))\cdot D\sigma_1(v)=1$$

b) We remark that if $\gamma:[a,b]\to \boldsymbol{B}_2^2$ and $\sigma:[b,c]\to \boldsymbol{B}_2^2$ there take place the formulas that are symbolized by:

$$\int_a^c = \int_a^b \oplus \int_b^c, \quad \int_a^{c\,*} = \int_a^{b\,*} \oplus \int_b^{c\,*}$$

c) We have $\gamma,\gamma^-:[a,b]\to \boldsymbol{B}_2^2$ and the decreasing homeomorphism $h:[a,b]\to[a,b]$ that is given by $h(t)=a+b-t$ and makes commutative the diagram:

$$\begin{tikzcd}
{[a,b]} \arrow[rr, "h"] \arrow[dr, "\gamma"'] & & {[a,b]} \arrow[dl, "\gamma^{-}"] \\
& \boldsymbol{B}_2^2 &
\end{tikzcd}$$

Analoguely to the situation from item a), we can show that for any $u, v \in (a,b)$, $v = h(u)$, it is true for example the equation:

$$\omega_1(\gamma(u)) \cdot D^* \gamma_1(u) = \omega_1(\gamma^{-}(v)) \cdot D\gamma_1^{-}(v)$$

This is also the case if $u = a$, $v = b$ and we can write:

$$\int_a^b \omega_1(\gamma) \cdot D^* \gamma_1 = \underset{u \in [a,b)}{\Xi} \omega_1(\gamma(u)) \cdot D^* \gamma_1(u) = \underset{u \in (a,b)}{\Xi} \omega_1(\gamma(u)) \cdot D^* \gamma_1(u) \oplus \omega_1(\gamma(a)) \cdot D^* \gamma_1(a) =$$

$$= \underset{v \in (a,b)}{\Xi} \omega_1(\gamma^{-}(v)) \cdot D\gamma_1^{-}(v) \oplus \omega_1(\gamma^{-}(b)) \cdot D\gamma_1^{-}(b)$$

$$= \underset{v \in (a,b]}{\Xi} \omega_1(\gamma^{-}(v)) \cdot D\gamma_1^{-}(v) = \int_a^{b\,*} \omega_1(\gamma^{-}) \cdot D\gamma_1^{-}$$

The other equations may be proved similarly.

11. Let $\omega : \boldsymbol{B}_2^2 \to F(\boldsymbol{B}_2^2, \boldsymbol{B}_2)$ be a Boolean form. It is called *exact* if there exists $f : \boldsymbol{B}_2^2 \to \boldsymbol{B}_2$ with the property that:

$$\forall x \in \boldsymbol{B}_2^2, \partial_1 f(x) = \omega_1(x), \partial_2 f(x) = \omega_2(x), \partial_{12}^2 f(x) = \omega_3(x)$$

where ∂ is the notation for the *Boolean derivative*

$$\partial_1 f(x_1, x_2) = f(x_1 \oplus 1, x_2) \oplus f(x_1, x_2)$$

$$\partial_2 f(x_1, x_2) = f(x_1, x_2 \oplus 1) \oplus f(x_1, x_2)$$

$$\partial_{12}^2 f(x_1, x_2) = \partial_1(\partial_2 f(x_1, x_2)) = \partial_2(\partial_1 f(x_1, x_2))$$

12. **Remark** The functions $f : \boldsymbol{B}_2^2 \to \boldsymbol{B}_2$ may be put under the form

$$f(x_1, x_2) = a \oplus b \cdot x_1 \oplus c \cdot x_2 \oplus d \cdot x_1 \cdot x_2$$

where $a, b, c, d \in \boldsymbol{B}_2$ and then, if ω is exact, it is given – after following an identification procedure –by:

$$\omega_1(x_1, x_2) = b \oplus d \cdot x_2$$

$$\omega_2(x_1, x_2) = c \oplus d \cdot x_1$$

$$\omega_3(x_1, x_2) = d$$

It is seen the validity of the formula:

$$\partial_2 \omega_1(x_1, x_2) = \partial_1 \omega_2(x_1, x_2) (= d)$$

recalling the theory of the real functions (the closed real differentiable forms).

Conversely, if $\omega_1, \omega_2, \omega_3$ are like above, then ω is exact.

13. **Theorem** The next statements are equivalent:

a) ω is exact

b) for any closed path $\gamma : [a,b] \to \boldsymbol{B}_2^2$, we have

$$\int_\gamma \omega = 0$$

c) for any paths $\gamma:[a,b]\to \boldsymbol{B}_2^2, \sigma:[c,d]\to \boldsymbol{B}_2^2$ with the same extremes $\gamma(a)=\sigma(c), \gamma(b)=\sigma(d)$ we have:

$$\int_\gamma \omega = \int_\sigma \omega$$

Proof a) $\Rightarrow$ b) There are two lemmas that we need.

14. **Lemma** Being given $f:\boldsymbol{B}_2^2 \to \boldsymbol{B}_2$ and the function $\gamma:(a,b]\to \boldsymbol{B}_2^2$ with left limits, there takes place the formula:

$$Df(\gamma_1(t),\gamma_2(t)) = \partial_1 f(\gamma_1(t),\gamma_2(t))\cdot D\gamma_1(t)\oplus$$
$$\oplus \partial_2 f(\gamma_1(t),\gamma_2(t))\cdot D\gamma_2(t)\oplus \partial_{12}^2 f(\gamma_1(t),\gamma_2(t))\cdot D\gamma_1(t)\cdot D\gamma_2(t)$$

This fact may be seen from the following equalities, that are valid for any $a,b,c,d,x_1,x_2,y_1,y_2 \in \boldsymbol{B}_2$:

$$f(x_1,x_2) = a\oplus b\cdot x_1 \oplus c\cdot x_2 \oplus d\cdot x_1\cdot x_2$$
$$\partial_1 f(x_1,x_2) = b\oplus d\cdot x_2, \partial_2 f(x_1,x_2) = c\oplus d\cdot x_1$$
$$\partial_{12}^2 f(x_1,x_2) = d$$
$$\Rightarrow f(x_1,x_2)\oplus f(y_1,y_2) = \partial_1 f(x_1,x_2)\cdot(y_1\oplus x_1)\oplus \partial_2 f(x_1,x_2)\cdot(y_2\oplus x_2)\oplus$$
$$\oplus \partial_{12}^2 f(x_1,x_2)\cdot(y_1\oplus x_1)\cdot(y_2\oplus x_2)$$

(identity easy to remark). Furthermore, by considering

$$(x_1,x_2) = (\gamma_1(t),\gamma_2(t))$$
$$(y_1,y_2) = (\gamma_1(t-0),\gamma_2(t-0))$$

the statement of the lemma results.

15. **Lemma** Let $x:[a,b]\to \boldsymbol{B}_2$ with left limits and right limits. It is true the formula of Leibniz-Newton:

$$\int_a^b D^* x \oplus \int_a^{b\,*} Dx = x(a)\oplus x(b)$$

Proof Let the points

$$a = t_0 < t_1 < ... < t_n = b$$

like in the proof of lemma 5,defined by the property that the values $c_0,c_1,...,c_n,d_1,...,d_n \in \boldsymbol{B}_2$ exist so that

$$x(t_i) = c_i, i = \overline{0,n}$$
$$\forall \xi \in (t_i,t_{i+1}), x(\xi) = d_{i+1}, i = \overline{0,n-1}$$

We see that

$$\int_a^b D^* x \oplus \int_a^{b\,*} Dx = (c_0\oplus d_1)\oplus(c_1\oplus d_2)\oplus ...\oplus(c_{n-1}\oplus d_n)\oplus$$
$$\oplus(d_1\oplus c_1)\oplus(d_2\oplus c_2)\oplus ...\oplus(d_n\oplus c_n) = c_0\oplus c_n = x(a)\oplus x(b)$$

We may proceed now to prove the implication a) $\Rightarrow$ b) and from the hypothesis there exists a function $f:\boldsymbol{B}_2^2 \to \boldsymbol{B}_2$ with the property that

$$\forall x\in \boldsymbol{B}_2^2, \partial_1 f(x) = \omega_1(x), \partial_2 f(x) = \omega_2(x), \partial_{12}^2 f(x) = \omega_3(x)$$

We take some path $\gamma:[a,b]\to \boldsymbol{B}_2^2$ with the property that $\gamma(a)=\gamma(b)$. We may write, by using lemma 14 and its right dual:

$$Df(\gamma_1(t),\gamma_2(t))=\partial_1 f(\gamma_1(t),\gamma_2(t))\cdot D\gamma_1(t)\oplus$$
$$\oplus\partial_2 f(\gamma_1(t),\gamma_2(t))\cdot D\gamma_2(t)\oplus\partial_{12}^2 f(\gamma_1(t),\gamma_2(t))\cdot D\gamma_1(t)\cdot D\gamma_2(t)=$$
$$=\omega_1(\gamma_1(t),\gamma_2(t))\cdot D\gamma_1(t)\oplus\omega_2(\gamma_1(t),\gamma_2(t))\cdot D\gamma_2(t)\oplus\omega_3(\gamma_1(t),\gamma_2(t))\cdot D\gamma_1(t)\cdot D\gamma_2(t)$$
$$D^* f(\gamma_1(t),\gamma_2(t))=\omega_1(\gamma_1(t),\gamma_2(t))\cdot D^*\gamma_1(t)\oplus\omega_2(\gamma_1(t),\gamma_2(t))\cdot D^*\gamma_2(t)\oplus$$
$$\oplus\omega_3(\gamma_1(t),\gamma_2(t))\cdot D^*\gamma_1(t)\cdot D^*\gamma_2(t)$$

and from lemma 15:

$$\int_\gamma\omega=\int_a^b D^* f(\gamma)\oplus\int_a^{b\,*} Df(\gamma)=f(\gamma(a))\oplus f(\gamma(b))=0$$

b) $\Rightarrow$ c) Let us take $\gamma:[c,d]\to\boldsymbol{B}_2^2$ and $\sigma:[c',d']\to\boldsymbol{B}_2^2$ two paths satisfying the conditions:

$$\gamma(c)=\sigma(c'),\gamma(d)=\sigma(d')$$

We fix $a<b$ and $a'\in(a,b)$ and choose the increasing homeomorphism $h_1:[a,a']\to[c',d']$ respectively the decreasing homeomorphism $h_2:[a',b]\to[c',d']$ that give the definitions

$$\alpha_1:[a,a']\to\boldsymbol{B}_2^2,\ \alpha_1\overset{def}{=}\gamma\circ h_1$$
$$\alpha_2:[a',b]\to\boldsymbol{B}_2^2,\ \alpha_2\overset{def}{=}\sigma\circ h_2$$

The reunion $\alpha=\alpha_1\vee\alpha_2:[a,b]\to\boldsymbol{B}_2^2$ makes sense because

$$\alpha_1(a')=(\gamma\circ h_1)(a')=\gamma(d)=\sigma(d')=(\sigma\circ h_2)(a')=\alpha_2(a')$$

and it is a closed path

$$\alpha(a)=\alpha_1(a)=(\gamma\circ h_1)(a)=\gamma(c)=\sigma(c')=(\sigma\circ h_2)(b)=\alpha_2(b)=\alpha(b)$$

By making use of the hypothesis, as well as of the fact that $\alpha_1\sim\gamma,\alpha_2^-\sim\sigma,$ we have:

$$0=\int_{\alpha_1\vee\alpha_2}\omega=\int_{\alpha_1}\omega\oplus\int_{\alpha_2}\omega=\int_{\alpha_1}\omega\oplus\int_{\alpha_2^-}\omega=\int_\gamma\omega\oplus\int_\sigma\omega$$

c) $\Rightarrow$ a) Let us put ω under he form:

$$\omega_i(x_1,x_2)=a_i\oplus b_i\cdot x_1\oplus c_i\cdot x_2\oplus d_i\cdot x_1\cdot x_2$$

where $a_i,b_i,c_i,d_i\in\boldsymbol{B}_2, i=\overline{1,3}$ and we show, by choosing convenient pairs of paths $\gamma,\sigma:[a,b]\to\boldsymbol{B}_2^2$ with the same extremes: $\gamma(a)=\sigma(a),\ \gamma(b)=\sigma(b)$ that ω is of the same form like the one indicated in Remark 12. We shall use the notation $\chi_{(\cdot)}:[a,b]\to\boldsymbol{B}_2$ for the characteristic function and let $\tau\in(a,b)$;

$$\gamma=(\chi_{(\tau,b]},0)$$
$$\sigma=(\chi_{[\tau,b]},0)$$

give, because the only non-zero derivatives are $D^*\gamma_1=\chi_{\{\tau\}},D\sigma_1=\chi_{\{\tau\}}$,

$$\int_\gamma\omega=\omega_1(\gamma(\tau))=a_1,\int_\sigma\omega=\omega_1(\sigma(\tau))=a_1\oplus b_1$$

so that $a_1=a_1\oplus b_1$ and $b_1=0$.
This reasoning was written in table 1.

γ	σ	The equation	The conclusions	Remarks
$(\chi_{(\tau,b]},0)$	$(\chi_{[\tau,b]},0)$	$a_1 = a_1 \oplus b_1$	$b_1 = 0$	(1)
$(0,\chi_{(\tau,b]})$	$(0,\chi_{[\tau,b]})$	$a_2 = a_2 \oplus c_2$	$c_2 = 0$	(2)
$(\chi_{(\tau,b]},1)$	$(\chi_{[\tau,b]},1)$	$a_1 \oplus c_1 = a_1 \oplus b_1 \oplus c_1 \oplus d_1$	$d_1 = 0$	(3) from (1)
$(1,\chi_{(\tau,b]})$	$(1,\chi_{[\tau,b]})$	$a_2 \oplus b_2 = a_2 \oplus b_2 \oplus c_2 \oplus d_2$	$d_2 = 0$	(4) from (2)
$(\chi_{(\tau,b]},\chi_{(\tau,b]})$	$(\chi_{[\tau,b]},\chi_{[\tau,b]})$	$a_1 \oplus a_2 \oplus a_3 = a_1 \oplus c_1 \oplus a_2 \oplus b_2 \oplus a_3 \oplus b_3 \oplus c_3 \oplus d_3$	$c_1 \oplus b_2 \oplus b_3 \oplus c_3 \oplus d_3 = 0$	(5)
$(\chi_{[a,\tau]},\chi_{[a,\tau]})$	$(\chi_{[a,\tau)},\chi_{[a,\tau]})$	$a_1 \oplus c_1 \oplus a_2 \oplus b_2 \oplus a_3 \oplus b_3 \oplus c_3 \oplus d_3 = a_1 \oplus c_1 \oplus a_2$	$b_2 \oplus a_3 \oplus b_3 \oplus c_3 \oplus d_3 = 0$	(6)
			$c_1 = a_3$	(7) from (5), (6)
$(\chi_{[a,\tau]},\chi_{[a,\tau]})$	$(\chi_{[a,\tau]},\chi_{[a,\tau)})$	$a_1 \oplus c_1 \oplus a_2 \oplus b_2 \oplus a_3 \oplus b_3 \oplus c_3 \oplus d_3 = a_1 \oplus a_2 \oplus b_2$	$c_1 \oplus a_3 \oplus b_3 \oplus c_3 \oplus d_3 = 0$	(8)
			$b_2 = a_3$	(9) from (5),(8)
			$c_1 = b_2$	(10) from (7),(9)
			$b_3 \oplus c_3 \oplus d_3 = 0$	(11) from (6),(9)
$(\chi_{[a,\tau)},\chi_{[\tau,b]})$	$(\chi_{[a,\tau]},\chi_{(\tau,b]})$	$a_1 \oplus c_1 \oplus a_2 \oplus a_3 \oplus c_3 = a_1 \oplus a_2 \oplus b_2 \oplus a_3 \oplus b_3$	$c_1 \oplus c_3 \oplus b_2 \oplus b_3 = 0$	(12)
			$b_3 = c_3$	(13) from (10),(12)
			$d_3 = 0$	(14) from (11),(13)
$(\chi_{[a,\tau)\vee(\nu,b]},\chi_{[\tau,b]})$	$(1,\chi_{(\tau,b]})$	$a_1 \oplus c_1 \oplus a_2 \oplus c_2 \oplus a_3 \oplus c_3 \oplus a_1 \oplus c_1 = a_2 \oplus b_2$	$a_3 \oplus c_3 \oplus b_2 = 0$	(15) $a < \tau < \nu < b$ from (2)
			$c_3 = b_3 = 0$	(16) from (9),(13),(15)

Table 1