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BINARY SIGNALS: THE SET OF THE PERIODS OF A PERIODIC POINT

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Abstract. The asynchronous systems are the discrete time and real time models of the asynchronous circuits from digital electrical engineering. The functions that these systems work with, called (binary) signals, are the models of the electrical signals. In our paper we prove that the sets of periods of the periodic points of the non constant signals are of the form $\{p,2p,3p,...\}, p \geq 1$ (discrete time) and $\{T,2T,3T,...\}, T > 0$ (real time).

Key words: binary signal; period; prime period.

1. Introduction

The asynchronous systems are the models of the asynchronous circuits (digital electrical circuits) and the functions representing their inputs and states are called (binary) signals. An important special case consists in systems that are generated by Boolean functions $\Phi:\{0,1\}^n \to \{0,1\}^n$ that iterate (like the dynamical systems), but the iterations happen on some coordinates $\Phi_1,...,\Phi_n$ only, not on all the coordinates (unlike the dynamical systems). Other classes of

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asynchronous systems are defined by equations and, in the most general case, by multi-valued functions (Vlad, 2007).

In order to study the periodicity of the asynchronous systems, we need to study the periodicity of the (values of the) signals first. Our present aim is to show that, when the signals are not constant, the sets of periods of their points are of the form $\{p,2p,3p,...\}, p \geq 1$ in discrete time and $\{T,2T,3T,...\}, T > 0$ in real time.

2. Preliminaries. Signals

The set $\boldsymbol{B} = \{0,1\}$ is a field relative to the modulo 2 sum $'\oplus'$ and the product $'\cdot'$ and also a topological space relative to the discrete topology. Natural structures are induced on $\boldsymbol{B}^n, n \geq 1$.

Notation 1. We have the notation $\chi_A : \boldsymbol{R} \to \boldsymbol{B}$ for the characteristic function of the set $A \subset \boldsymbol{R} : \forall t \in \boldsymbol{R}$,

$$\chi_A(t) = \begin{cases} 1, if\ t \in A, \\ 0, otherwise. \end{cases}$$

Notation 2. We denote $\boldsymbol{N}_{\_} = \{-1,0,1,...\}$.

Definition 3. The *discrete time signals* are by definition the functions $\hat{x} : \boldsymbol{N}_{\_} \to \boldsymbol{B}^n$. Their set is denoted by $\hat{S}^{(n)}$.

The *continuous time signals* are the functions $x : \boldsymbol{R} \to \boldsymbol{B}^n$ of the form $\forall t \in \boldsymbol{R}$,

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

where $\mu \in \boldsymbol{B}^n$ and $t_k \in \boldsymbol{R}, k \in \boldsymbol{N}$ is strictly increasing and unbounded from above. Their set is denoted by $S^{(n)}$. μ is usually denoted by $x(-\infty + 0)$ and is called the *initial value* of x.

Remark 4. Let $x \in S^{(n)}$ and $t \in \boldsymbol{R}$ arbitrary. We have the properties of existence of the left limit $x(t-0) \in \boldsymbol{B}^n : \exists \varepsilon > 0, \forall \xi \in (t-\varepsilon, t), x(\xi) = x(t-0)$, of the right limit $x(t+0) \in \boldsymbol{B}^n : \exists \varepsilon > 0, \forall \xi \in (t, t+\varepsilon), x(\xi) = x(t+0)$ and of right continuity: $x(t) = x(t+0)$. The last property will be used in the paper under the form $\exists \varepsilon > 0, \forall \xi \in [t, t+\varepsilon), x(\xi) = x(t)$. These things are discussed for example in (Vlad, 2007).

Lemma 5. Let $x \in S^{(n)}$ and the sequence $T_k \in \boldsymbol{R}, k \in \boldsymbol{N}$ that is strictly decreasing convergent to $T \in \boldsymbol{R}$. Then $\exists N \in \boldsymbol{N}, \forall k \geq N$,

$$x(T_k - 0) = x(T_k) = x(T). \quad (1)$$

Proof. Some $\varepsilon > 0$ exists with the property that

$$\forall\xi \in [T, T+\varepsilon), x(\xi) = x(T) . \tag{2}$$

As $T_k \to T$ strictly decreasing, $N_\varepsilon \in \boldsymbol{N}$ exists such that

$$\forall k \geq N_\varepsilon, T < T_k < T + \varepsilon . \tag{3}$$

We fix an arbitrary $k \geq N_\varepsilon$. If we take $\varepsilon' \in (0, T_k - T)$, we have

$$T - T_k < -\varepsilon' < 0 . \tag{4}$$

We add T_k to the terms of (4) and we obtain, taking into account (3) also:

$$T < T_k - \varepsilon' < T_k < T + \varepsilon . \tag{5}$$

We conclude on one hand that

$$\forall\xi \in (T_k - \varepsilon', T_k), x(\xi) \overset{(2),(5)}{=} x(T) ,$$

thus

$$x(T_k - 0) = x(T) \tag{6}$$

and on the other hand that

$$x(T_k) \overset{(2),(5)}{=} x(T) . \tag{7}$$

By comparing (6) with (7) we infer (1).

Definition 6. The sets

$$Or(\hat{x}) = \{\hat{x}(k) \mid k \in \boldsymbol{N}_{\_}\} ,$$

$$Or(x) = \{x(t) \mid t \in \boldsymbol{R}\}$$

are called the *orbits* of $\hat{x}, x$.

Notation 7. For $\hat{x} \in \hat{S}^{(n)}, x \in S^{(n)}$ and $\mu \in Or(\hat{x}), \nu \in Or(x)$, we use the notations

$$\boldsymbol{T}_\mu^{\hat{x}} = \{k \mid k \in \boldsymbol{N}_{\_}, \hat{x}(k) = \mu\} ,$$

$$\boldsymbol{T}_\nu^{x} = \{t \mid t \in \boldsymbol{R}, x(t) = \mu\} .$$

Lemma 8. Let $\mu \in Or(x)$ and $t' \in \boldsymbol{R}$. If $(-\infty, t'] \subset \boldsymbol{T}_{x(-\infty+0)}^{x}$, then $\boldsymbol{T}_\mu^{x} \cap [t', \infty) \neq \varnothing$.

Proof. We have two possibilities.

Case $\mu = x(-\infty + 0)$, when $t' \in \boldsymbol{T}_\mu^{x}$, makes that $\boldsymbol{T}_\mu^{x} \cap [t', \infty) \neq \varnothing$ be true.

Case $\mu \neq x(-\infty + 0)$, when $\boldsymbol{T}_\mu^{x} \cap (-\infty, t'] = \varnothing$, $\boldsymbol{T}_\mu^{x} \neq \varnothing$ make that $\boldsymbol{T}_\mu^{x} \subset (t', \infty)$, thus $\boldsymbol{T}_\mu^{x} \cap [t', \infty) \neq \varnothing$.

3. Periodic Points

Definition 9. We consider the signals $\hat{x} \in \hat{S}^{(n)}, x \in S^{(n)}$.

Let $\mu \in Or(\hat{x})$ and $p \geq 1$. If

$$\forall k \in \boldsymbol{T}_{\mu}^{\hat{x}}, \{k + zp \mid z \in \boldsymbol{Z}\} \cap \boldsymbol{N}_{\_} \subset \boldsymbol{T}_{\mu}^{\hat{x}}, \tag{8}$$

we say that μ is *periodic*, with the *period* p. The least p that fulfills (8) is called the *prime period* of μ. The set of the periods of μ is denoted with $P_{\mu}^{\hat{x}}$.

Let $\mu \in Or(x)$ and $T > 0, t' \in \boldsymbol{R}$ such that

$$(-\infty, t'] \subset \boldsymbol{T}_{x(-\infty+0)}^{x}, \tag{9}$$

$$\forall t \in \boldsymbol{T}_{\mu}^{x} \cap [t', \infty), \{t + zT \mid z \in \boldsymbol{Z}\} \cap [t', \infty) \subset \boldsymbol{T}_{\mu}^{x}. \tag{10}$$

Then μ is called *periodic*, with the *period* T. The least T with the property that t' exists such that (9), (10) are fulfilled is called the *prime period* of μ. The set of the periods of μ is denoted with P_{μ}^{x}.

Lemma 10. Let $x \in S^{(n)}, \mu \in Or(x), T > 0, t' < t''$ such that

$$\forall t \in \boldsymbol{T}_{\mu}^{x} \cap [t', \infty), \{t + zT \mid z \in \boldsymbol{Z}\} \cap [t', \infty) \subset \boldsymbol{T}_{\mu}^{x}, \tag{11}$$

$$x(t''-0) \neq x(t'') = \mu. \tag{12}$$

Then

$$x(t''+T-0) \neq x(t''+T) = \mu. \tag{13}$$

Proof. From (11) and $x(t'') = \mu$, since $t' < t''$, we infer $x(t''+T) = \mu$.

Some $\varepsilon' > 0, \varepsilon'' > 0$ exist having the property that

$$\forall \xi \in (t''-\varepsilon', t''), x(\xi) = x(t''-0), \tag{14}$$

$$\forall \xi \in (t''+T-\varepsilon'', t''+T), x(\xi) = x(t''+T-0) \tag{15}$$

and for any $\varepsilon \in (0, \min\{\varepsilon', \varepsilon'', t''-t'\})$ we can write that

$$(t''-\varepsilon, t'') \subset [t', \infty), \tag{16}$$

$$(t''-\varepsilon, t'') \subset (t''-\varepsilon', t''), \tag{17}$$

$$(t''+T-\varepsilon, t''+T) \subset (t''+T-\varepsilon'', t''+T) \tag{18}$$

hold. We take now an arbitrary, fixed $\xi \in (t''-\varepsilon, t'')$. We infer

$$x(t''+T-0) \overset{(15),(18)}{=} x(\xi+T) \overset{(11),(16)}{=} x(\xi) =$$

$$\overset{(14),(17)}{=} x(t''-0) \overset{(12)}{\neq} x(t'') \overset{(12)}{=} \mu \overset{(11)}{=} x(t''+T),$$

thus (13) is proved.

4. Sums, Differences and Multiples of Periods

Theorem 11. The signals $\hat{x}\in\hat{S}^{(n)}, x\in S^{(n)}$ are considered.

a) Let $p,p'\geq 1,\mu\in Or(\hat{x})$ and we ask that

$$\forall k\in \boldsymbol{T}_{\mu}^{\hat{x}},\{k+zp\mid z\in \boldsymbol{Z}\}\cap \boldsymbol{N}\_ \subset \boldsymbol{T}_{\mu}^{\hat{x}}, \tag{19}$$

$$\forall k\in \boldsymbol{T}_{\mu}^{\hat{x}},\{k+zp'\mid z\in \boldsymbol{Z}\}\cap \boldsymbol{N}\_ \subset \boldsymbol{T}_{\mu}^{\hat{x}} \tag{20}$$

hold. We have $p+p'\geq 1$,

$$\forall k\in \boldsymbol{T}_{\mu}^{\hat{x}},\{k+z(p+p')\mid z\in \boldsymbol{Z}\}\cap \boldsymbol{N}\_ \subset \boldsymbol{T}_{\mu}^{\hat{x}},$$

and if $p>p'$, then $p-p'\geq 1$,

$$\forall k\in \boldsymbol{T}_{\mu}^{\hat{x}},\{k+z(p-p')\mid z\in \boldsymbol{Z}\}\cap \boldsymbol{N}\_ \subset \boldsymbol{T}_{\mu}^{\hat{x}}.$$

b) Let $T,T'>0,t',t''\in \boldsymbol{R},\mu\in Or(x)$ be arbitrary with

$$(-\infty,t']\subset \boldsymbol{T}_{x(-\infty+0)}^{x},$$

$$\forall t\in \boldsymbol{T}_{\mu}^{x}\cap[t',\infty),\{t+zT\mid z\in \boldsymbol{Z}\}\cap[t',\infty)\subset \boldsymbol{T}_{\mu}^{x}, \tag{21}$$

$$(-\infty,t'']\subset \boldsymbol{T}_{x(-\infty+0)}^{x}, \tag{22}$$

$$\forall t\in \boldsymbol{T}_{\mu}^{x}\cap[t'',\infty),\{t+zT'\mid z\in \boldsymbol{Z}\}\cap[t'',\infty)\subset \boldsymbol{T}_{\mu}^{x} \tag{23}$$

fulfilled. We have on one hand that $T+T'>0$ and $t_1\in \boldsymbol{R}$ exists with

$$(-\infty,t_1]\subset \boldsymbol{T}_{x(-\infty+0)}^{x},$$

$$\forall t\in \boldsymbol{T}_{\mu}^{x}\cap[t_1,\infty),\{t+z(T+T')\mid z\in \boldsymbol{Z}\}\cap[t_1,\infty)\subset \boldsymbol{T}_{\mu}^{x}$$

and on the other hand that $T>T'$ implies $T-T'>0$ and $t_2\in \boldsymbol{R}$ exists with

$$(-\infty,t_2]\subset \boldsymbol{T}_{x(-\infty+0)}^{x},$$

$$\forall t\in \boldsymbol{T}_{\mu}^{x}\cap[t_2,\infty),\{t+z(T-T')\mid z\in \boldsymbol{Z}\}\cap[t_2,\infty)\subset \boldsymbol{T}_{\mu}^{x}.$$

Proof. a) We prove the second implication. We take some arbitrary, fixed $k\in \boldsymbol{T}_{\mu}^{\hat{x}},z\in \boldsymbol{Z}$ such that $k+z(p-p')\geq -1$ and we have the following possibilities:

Case $z<0$

We obtain in succession $k-zp'\geq -1$, $k-zp' \overset{(20)}{\in} \boldsymbol{T}_{\mu}^{\hat{x}}$, $k-zp'+zp \overset{hyp}{\geq} -1$, $k+z(p-p') \overset{(19)}{\in} \boldsymbol{T}_{\mu}^{\hat{x}}$.

Case $z=0$

$k=k+z(p-p')\in \boldsymbol{T}_{\mu}^{\hat{x}}$ trivially.

Case $z>0$

We have $k+zp\geq -1$, $k+zp\overset{(19)}{\in}\boldsymbol{T}_{\mu}^{\hat{x}}$, $k+zp-zp'\overset{hyp}{\geq}-1$, $k+z(p-p')\overset{(20)}{\in}\boldsymbol{T}_{\mu}^{\hat{x}}$.

b) We prove the first implication. We notice that for $t_1=\max\{t',t''\}$ the following statements

$$(-\infty,t_1]\subset\boldsymbol{T}_{x(-\infty+0)}^{x},\tag{24}$$

$$\forall t\in\boldsymbol{T}_{\mu}^{x}\cap[t_1,\infty),\{t+zT)\,|\,z\in\boldsymbol{Z}\}\cap[t_1,\infty)\subset\boldsymbol{T}_{\mu}^{x},\tag{25}$$

$$\forall t\in\boldsymbol{T}_{\mu}^{x}\cap[t_1,\infty),\{t+zT')\,|\,z\in\boldsymbol{Z}\}\cap[t_1,\infty)\subset\boldsymbol{T}_{\mu}^{x}\tag{26}$$

hold. Indeed, in order to see this fact we suppose without loosing the generality that $t'<t'',t_1=t''$ are true. Then (24) coincides with (22) and (25), (26) coincide with

$$\forall t\in\boldsymbol{T}_{\mu}^{x}\cap[t'',\infty),\{t+zT)\,|\,z\in\boldsymbol{Z}\}\cap[t'',\infty)\subset\boldsymbol{T}_{\mu}^{x}\tag{27}$$

and (23). From Lemma 8 we get $\boldsymbol{T}_{\mu}^{x}\cap[t'',\infty)\neq\varnothing$. Let $t\in\boldsymbol{T}_{\mu}^{x}\cap[t'',\infty)$ and $z\in\boldsymbol{Z}$ arbitrary such that $t+zT\geq t''$. Then $t\in\boldsymbol{T}_{\mu}^{x}\cap[t',\infty)$ and $t+zT\geq t'$, thus $t+zT\overset{(21)}{\in}\boldsymbol{T}_{\mu}^{x}$, (27) holds, and we can use (24),...,(26).

Let now $t\in\boldsymbol{T}_{\mu}^{x}\cap[t_1,\infty)$, $z\in\boldsymbol{Z}$ be arbitrary, fixed such that $t+z(T+T')\geq t_1$.

Case $z<0$

We have in succession $t+zT\geq t+z(T+T')\overset{hyp}{\geq}t_1$, $t+zT\overset{(25)}{\in}\boldsymbol{T}_{\mu}^{x}$, $t+z(T+T')\overset{(26)}{\in}\boldsymbol{T}_{\mu}^{x}$.

Case $z=0$

We infer $t=t+z(T+T')\in\boldsymbol{T}_{\mu}^{x}$.

Case $z>0$

We have $t+zT\geq t\geq t_1$, $t+zT\overset{(25)}{\in}\boldsymbol{T}_{\mu}^{x}$, $t+z(T+T')\overset{hyp}{\geq}t_1$, $t+z(T+T')\overset{(26)}{\in}\boldsymbol{T}_{\mu}^{x}$.

Corollary 12. a) If $p\in P_{\mu}^{\hat{x}}$, then $\{p,2p,3p,...\}\subset P_{\mu}^{\hat{x}}$.

b) If $T\in P_{\mu}^{x}$, then $\{T,2T,3T,...\}\subset P_{\mu}^{x}$.

Proof. These statements result from Theorem 11.

5. The Set of the Periods

Theorem 13. a) Let the signal $\hat{x} \in \hat{S}^{(n)}$ and $\mu \in Or(\hat{x})$. We ask that μ is a periodic point of $\hat{x}$. Then some $\widetilde{p} \geq 1$ exists such that

$$P_{\mu}^{\hat{x}} = \{\widetilde{p}, 2\widetilde{p}, 3\widetilde{p}, ...\}.$$

b) We suppose that the signal $x \in S^{(n)}$ is not constant and we take some $\mu \in Or(x)$. We ask that μ is a periodic point of x. Then there is $\widetilde{T} > 0$ such that

$$P_{\mu}^{x} = \{\widetilde{T}, 2\widetilde{T}, 3\widetilde{T}, ...\}.$$

Proof. a) We denote with $\widetilde{p}$ the least element of $P_{\mu}^{\hat{x}}$. From Corollary 12 we have the inclusion $\{\widetilde{p}, 2\widetilde{p}, 3\widetilde{p}, ...\} \subset P_{\mu}^{\hat{x}}$. We show that $P_{\mu}^{\hat{x}} \subset \{\widetilde{p}, 2\widetilde{p}, 3\widetilde{p}, ...\}$. We presume against all reason that this is not true, i.e. that some $p' \in P_{\mu}^{\hat{x}} - \{\widetilde{p}, 2\widetilde{p}, 3\widetilde{p}, ...\}$ exists. In these circumstances we have the existence of $k_1 \geq 1$ with $k_1\widetilde{p} < p' < (k_1+1)\widetilde{p}$. We infer that $1 \leq p' - k_1\widetilde{p} < \widetilde{p}$ and, from Theorem 11 and Corollary 12 we conclude that $p' - k_1\widetilde{p} \in P_{\mu}^{\hat{x}}$. We have obtained a contradiction with the fact that $\widetilde{p}$ is the least element of $P_{\mu}^{\hat{x}}$.

b) The proof is made in two steps.

b.1) We show first that $\min P_{\mu}^{x}$ exists. We suppose against all reason that this is not true, namely that a strictly decreasing sequence $T_k \in P_{\mu}^{x}, k \in \mathbf{N}$ exists that is convergent to $T = \inf P_{\mu}^{x}$. As x is not constant, some $t'' \in \mathbf{R}$ exists with the property

$$x(t''-0) \neq x(t'') = \mu. \tag{28}$$

The hypothesis states the existence $\forall k \in \mathbf{N}$, of $t_k^{'} \in \mathbf{R}$ with

$$(-\infty, t_k^{'}] \subset \boldsymbol{T}_{x(-\infty+0)}^{x}, \tag{29}$$

$$\forall t \in \boldsymbol{T}_{\mu}^{x} \cap [t_k^{'}, \infty), \{t + zT_k \mid z \in \mathbf{Z}\} \cap [t_k^{'}, \infty) \subset \boldsymbol{T}_{\mu}^{x}.$$

From (28), (29) we infer $t'' > t_k^{'}$ thus we can apply Lemma 10, wherefrom

$$\forall k \in \mathbf{N}, x(t''+T_k-0) \neq x(t''+T_k) = \mu. \tag{30}$$

We infer from Lemma 5 that $N \in \mathbf{N}$ exists with $\forall k \geq N$,

$$x(t''+T_k-0) = x(t''+T_k) = x(t''+T),$$

contradiction with (30). It has resulted that such a sequence $T_k, k \in \boldsymbol{N}$ does not exist, thus P_μ^x has a minimum that we denote by $\widetilde{T}$.

b.2) The inclusion $\{\widetilde{T}, 2\widetilde{T}, 3\widetilde{T}, ...\} \subset P_\mu^x$ results from Corollary 12, we prove the inclusion $P_\mu^x \subset \{\widetilde{T}, 2\widetilde{T}, 3\widetilde{T}, ...\}$. We suppose against all reason that some $T' \in P_\mu^x - \{\widetilde{T}, 2\widetilde{T}, 3\widetilde{T}, ...\}$ exists and let $k_1 \geq 1$ with the property $T' \in (k_1\widetilde{T}, (k_1+1)\widetilde{T})$. We infer that $0 < T' - k_1\widetilde{T} < \widetilde{T}$ and, from Theorem 11 and Corollary 12 we get $T' - k_1\widetilde{T} \in P_\mu^x$. We have obtained a contradiction, since $\widetilde{T}$ was defined to be the minimum of P_μ^x. $P_\mu^x = \{\widetilde{T}, 2\widetilde{T}, 3\widetilde{T}, ...\}$ holds.

Remark 14. We did not need to ask in the hypothesis of Theorem 13, item a) that $\hat{x}$ is not constant. When $\hat{x}$ is constant and equal with μ, then $\widetilde{p} = 1$ and $P_\mu^{\hat{x}} = \{1, 2, 3, ...\}$, thus item a) of the Theorem is still true. And if x is constant and equal with μ, then $P_\mu^x = (0, \infty)$.

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SEMNALE BINARE: MULȚIMEA PERIOADELOR UNUI PUNCT PERIODIC

(Rezumat)

Sistemele asincrone sunt modelele de timp discret şi timp real ale circuitelor asincrone din ingineria electronică digitală. Funcțiile cu care lucrează aceste sisteme, numite semnale (binare), sunt modelele semnalelor electrice. În această lucrare se demonstrează că mulțimile perioadelor punctelor periodice ale semnalelor neconstante sunt de forma $\{p, 2p, 3p, ...\}, p \geq 1$ pentru timpul discret şi $\{T, 2T, 3T, ...\}, T > 0$ pentru timpul real.