The decomposition of the regular asynchronous systems as parallel connection of regular asynchronous systems

Serban E. Vlad
Str. Zimbrului, Nr. 3, Bl. PB68, Ap. 11, 410430, Oradea, Romania, email: serban_e_vlad@yahoo.com

Abstract

The asynchronous systems are the non-deterministic models of the asynchronous circuits from the digital electrical engineering, where non-determinism is a consequence of the fact that modelling is made in the presence of unknown and variable parameters. Such a system is a multi-valued function f that assigns to an (admissible) input $u : \mathbf{R} \to \{0,1\}^m$ a set $f(u)$ of (possible) states $x : \mathbf{R} \to \{0,1\}^n$. When this assignment is defined by making use of a so-called generator function $\Phi : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}^n$, then the asynchronous system f is called regular. The generator function Φ acts in this asynchronous framework similarly with the next state function from a synchronous framework. The parallel connection of the asynchronous systems f' and f'' is the asynchronous system $(f'||f'')(u) = f'(u) \times f''(u)$. The purpose of the paper is to give the circumstances under which a regular asynchronous system f may be written as a parallel connection of regular asynchronous systems.

1 Introduction

The theory of modeling the asynchronous circuits from the digital electrical engineering has its origin in the switching (circuits) theory of the 50's and the 60's. By that time, researchers used the discrete time and mathematics seemed to be uncensored. After 1970, instead of switching theory (more exactly: instead of what we understand by switching theory), the analysis of these circuits is made in general by engineers that give approximate descriptions of the switching phenomena and draw pictures instead of writing equations. In fact the implicit suggestion given by the literature is that the real research is unpublished and a continuation of the switching theory, interrupted 40 years ago, is necessary. We have tried to do so and we have called this attempt the asynchronous systems theory.

The asynchronous systems theory makes use of $\mathbf{R} \to \{0,1\}$ functions (real time, binary values) that are not studied at all in literature (and have never been) as far as we know. The 'nice' $\mathbf{R} \to \{0,1\}$ functions are called signals, making us think of the digital electrical signals.

The modeling of the asynchronous circuits is made in the presence of unknown and variable parameters: the tension of the mains, the temperature, the delays that depend on technology. We could have used for this reason three valued signals $\mathbf{R} \to \{0,1,2\}$, but we prefer the present frame because of its algebraical advantages, the set $\{0,1\}$ is organized as a Boole algebra and as a field, unlike the set $\{0,1,2\}$. A certain price must be paid however, determinism is replaced by non-determinism, meaning that the systems working with $\mathbf{R} \to \{0,1,2\}$ functions may be considered to be input-output functions, but the systems working with $\mathbf{R} \to \{0,1\}$ functions must be considered input-output multi-valued functions. Thus an input is a function $u : \mathbf{R} \to \{0,1\}^m$, representing the cause and a state=output is a function $x : \mathbf{R} \to \{0,1\}^n$, representing the effect. The input is subject to certain constraints, not all the signals $\mathbf{R} \to \{0,1\}^m$ are allowed to be causes and this is why it is called admissible. The state is not unique, the system f assigns to u a family $f(u)$ of states $x \in f(u)$ that are called possible states; the unknown and the variable parameters that accompany the switching phenomena and f give us the certitude that when u is applied, an element of $f(u)$ will result.

The meaning of regularity is that of giving a special case of system, i.e. input-output multi-valued function, when a certain circuit is really modeled, since input-output multi-valued functions exist that model nothing. The regular asynchronous systems are these systems f that are 'generated' by a so-called 'generator function' $\Phi : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}^n$ (or 'network function' as Grigore Moisil called it; for Moisil a 'network' is a circuit identified with its model, the system).

The parallel connection of two systems f' and f'' is the system $(f'||f'')(u) = f'(u) \times f''(u)$ in which f' and f'' act independently on each other, but under a common input u. When f' is generated by Φ' and f'' is generated by Φ'', $f'||f''$ is a regular system also, which is generated by a function denoted by $\Phi'||\Phi''$. In $\Phi'||\Phi''$ some coordinates do not depend on other coordinates. This suggests the idea of considering the converse situation, when, from the fact that a system f is generated by the function Φ and in Φ some coordinates do not depend on other coordinates, we can infer the existence of some systems f', f'' and of some functions $\Phi' : \{0,1\}^{n'} \times \{0,1\}^m \to \{0,1\}^{n'}, \Phi'' : \{0,1\}^{n''} \times \{0,1\}^m \to \{0,1\}^{n''}$, such that $n'+n''=n$, f' is generated by Φ', f'' is generated by Φ'', $\Phi=\Phi'||\Phi''$ and $f=f'||f''$. Our present purpose is to study this possibility.

2 Preliminaries

Definition 1 *The set* $\mathbf{B}=\{0,1\}$ *endowed with the usual algebraical laws* $-$, $\cup$, $\cdot$, $\oplus$ *and with the order* $0<1$ *is called the* ***binary Boole algebra****.*

Definition 2 *The* ***characteristic function*** $\chi_A:\mathbf{R}\to\mathbf{B}$ *of the set* $A\subset\mathbf{R}$ *is defined by* $\forall t\in A$,

$$\chi_A(t)=\begin{cases}1, t\in A\\ 0, t\notin A\end{cases}.$$

Notation 3 *We denote by Seq the set of the sequences* $t_k\in\mathbf{R}$, $k\in\mathbf{N}$ *which are strictly increasing* $t_0<t_1<t_2<...$ *and unbounded from above. The elements of Seq will be denoted in general by* (t_k).

Definition 4 *The* ***signals*** *(or the* $n-$***signals****) are by definition the* $\mathbf{R}\to\mathbf{B}^n$ *functions of the form*

$$x(t)=\mu\cdot\chi_{(-\infty,t_0)}(t)\oplus x(t_0)\cdot\chi_{[t_0,t_1)}(t)\oplus...\oplus x(t_k)\cdot\chi_{[t_k,t_{k+1})}(t)\oplus... \quad (1)$$

where $t\in\mathbf{R}$, $\mu\in\mathbf{B}^n$ *and* $(t_k)\in Seq$. *The set of the signals is denoted by* $S^{(n)}$.

Definition 5 *In (1),* μ *is called the* ***initial value*** *of* x *and its usual notation is* $x(-\infty+0)$.

Definition 6 *The* ***Cartesian product*** *of the functions* $x':\mathbf{R}\to\mathbf{B}^{n'}$ *and* $x'':\mathbf{R}\to\mathbf{B}^{n''}$ *is the* $\mathbf{R}\to\mathbf{B}^{n'+n''}$ *function denoted by* (x',x''), $(x'(t),x''(t))$, $x'\times x''$ *or* $x'(t)\times x''(t)$ *which is defined by* $\forall i\in\{1,...,n'+n''\},\forall t\in\mathbf{R}$,

$$(x'(t),x''(t))_i=\begin{cases}x'_i(t), i\in\{1,...,n'\},\\ x''_i(t), i\in\{n'+1,n'+n''\}\end{cases}.$$

Notation 7 *For any set* M, *we denote with* $P^*(M)$ *the set of the non-empty subsets of* M.

Definition 8 *If* $X'\in P^*(S^{(n')})$ *and* $X''\in P^*(S^{(n'')})$, *then their* ***Cartesian product*** $X'\times X''\in P^*(S^{(n'+n'')})$ *is defined by*

$$X'\times X''=\{(x',x'')|x'\in X',x''\in X''\}.$$

Remark 9 *The signals model the electrical signals from the digital electrical engineering and* $\mathbf{R}$ *is the time set. The last two definitions of the Cartesian product replace the usual definition of* $x' \times x''$ *as a* $\mathbf{R} \times \mathbf{R} \to \mathbf{B}^{n'} \times \mathbf{B}^{n''}$ *function with* $x' \times x'' : \mathbf{R} \to \mathbf{B}^{n'+n''}$ *because the time is unique. We have suggested this idea in the notation of* $S^{(n)}$ *which is not* S^n*, because the Cartesian product* $\underbrace{S \times ... \times S}_{n} = S^{(n)}$ *is taken a little differently, in the sense of Definitions 6, 8. In such definitions we often identify* $\mathbf{B}^{n'} \times \mathbf{B}^{n''}$ *with* $\mathbf{B}^{n'+n''}$ *and* $S^{(n')} \times S^{(n'')}$ *with* $S^{(n'+n'')}$.

3 Asynchronous systems. Regularity

Definition 10 *An* ***asynchronous system*** *is a multi-valued function* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$. U *is called the* ***input set*** *and its elements* $u \in U$ *are called* ***(admissible) inputs****, while the functions* $x \in f(u)$ *are called* ***(possible) states****.*

Example 11 *The identity function* $1_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$ *is implemented in electrical engineering by the* ***delay circuit****. Such a circuit may be modelled by the system* $f : S^{(1)} \to P^*(S^{(1)})$*, called itself* ***delay****, which is given by the double inequality*

$$\bigcap_{\xi \in [t-\tau,t)} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-\tau,t)} u(\xi),$$

where $\tau > 0$ *and* $u, x \in S^{(1)}$*. We have for example*

$$f(\chi_{[0,\infty)}) = \{x | x \in S^{(1)}, \bigcap_{\xi \in [t-\tau,t)} \chi_{[0,\infty)}(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-\tau,t)} \chi_{[0,\infty)}(\xi)\}$$

$$= \{x | x \in S^{(1)}, \chi_{[\tau,\infty)}(t) \leq x(t) \leq \chi_{(0,\infty)}(t)\} = \{y \cdot \chi_{(0,\tau)} \oplus \chi_{[\tau,\infty)} | y \in S^{(1)}\},$$

thus when the input is $\chi_{[0,\infty)}$*, the state is* 0 *for* $t \leq 0$*, it is uncertain in the interval* $(0, \tau)$ *and it is equal with* 1 *for* $t \geq \tau$*. This system computes* $\forall \lambda \in \mathbf{B}$ *the value* $1_{\mathbf{B}}(\lambda) = \lambda$ *in at most* τ *time units.*

Definition 12 *Let be the system* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$*. The function* $\phi_0 : U \to P^*(\mathbf{B}^n)$ *defined by* $\forall u \in U$*,*

$$\phi_0(u) = \{x(-\infty + 0) | x \in f(u)\}$$

is called the ***initial state function*** *of* f*.*

Definition 13 *The binary sequence* $\alpha : \mathbf{N} \to \mathbf{B}^n, \alpha(k) = \alpha^k, k \in \mathbf{N}$ *is called* ***progressive*** *if* $\forall i \in \{1, ..., n\}$*, the set* $\{k | k \in \mathbf{N}, \alpha_i^k = 1\}$ *is infinite. The set of the progressive sequences is denoted by* Π_n*.*

Definition 14 *The function $\rho : \mathbf{R} \to \mathbf{B}^n$ is called* ***progressive*** *if it is of the form*

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{2}$$

with $\alpha \in \Pi_n$ and $(t_k) \in Seq$. The set of the progressive functions is denoted by P_n.

Definition 15 *Let be the function $\Phi : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$. For any $\nu \in \mathbf{B}^n$, we define the function $\Phi^\nu : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$ by $\forall \mu \in \mathbf{B}^n, \forall \lambda \in \mathbf{B}^m$,*

$$\Phi^\nu(\mu, \lambda) = (\overline{\nu_1} \cdot \mu_1 \oplus \nu_1 \cdot \Phi_1(\mu, \lambda), ..., \overline{\nu_n} \cdot \mu_n \oplus \nu_n \cdot \Phi_n(\mu, \lambda)).$$

Definition 16 *For all $\rho \in P_n$ like in (2), the function $\Phi^\rho : \mathbf{B}^n \times S^{(m)} \times \mathbf{R} \to \mathbf{B}^n$ is defined by $\forall \mu \in \mathbf{B}^n, \forall u \in S^{(m)}, \forall t \in \mathbf{R}$,*

$$\Phi^\rho(\mu, u, t) = \omega_{-1} \cdot \chi_{(-\infty, t_0)}(t) \oplus \omega_0 \cdot \chi_{[t_0, t_1)} \oplus ... \oplus \omega_k \cdot \chi_{[t_k, t_{k+1})} \oplus ...$$

where the family $\omega_k \in \mathbf{B}^n, k \in \mathbf{N} \cup \{-1\}$ is given by

$$\omega_{-1} = \mu,$$

$$\omega_{k+1} = \Phi^{\alpha^{k+1}}(\omega_k, u(t_{k+1})).$$

Definition 17 *The system $\Xi_\Phi : S^{(m)} \to P^*(S^{(n)})$ defined by $\forall u \in S^{(m)}$,*

$$\Xi_\Phi(u) = \{\Phi^\rho(\mu, u, \cdot) | \mu \in \mathbf{B}^n, \rho \in P_n\}$$

is called the ***universal regular asynchronous system*** *that is generated by the function Φ.*

Definition 18 *A system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ is called* ***regular*** *if Φ exists such that $\forall u \in U, f(u) \subset \Xi_\Phi(u)$. In this case the function Φ is called the* ***generator function*** *of f and we use to write $f \subset \Xi_\Phi$.*

Remark 19 *The asynchronous systems, as defined by us at Definition 10, represent a very general concept and such a multi-valued function may model nothing. The meaning of the regular asynchronous systems from Definition 18 is that of indicating a condition so that f really models a circuit, namely the circuit that implements the function Φ.*

For $\nu \in \mathbf{B}^n$, the function Φ^ν computes in Definition 15 the coordinates $\Phi_i(\mu, \lambda), i \in \{1, ..., n\}$ like this: if $\nu_i = 1$, then $\Phi_i^\nu(\mu, \lambda) = \Phi_i(\mu, \lambda)$ thus $\Phi_i(\mu, \lambda)$ is computed; if $\nu_i = 0$, then $\Phi_i^\nu(\mu, \lambda) = \mu_i$ thus $\Phi_i(\mu, \lambda)$ is not computed. The property of progress of the function ρ, Definitions 13, *14 assures in $\Phi^\rho(\mu, u, t)$ from Definition 16 the fact that the generator function Φ (which is not unique in general for some f) is computed in the following way: $\forall i \in \{1, ..., n\}, \forall t \in \mathbf{R}, \exists t_{k+1} > t$ such that $\Phi_i(\omega_k, u(t_{k+1}))$ is computed. This philosophy is different from the one of Example 11, where the value of the function $1_\mathbf{B}$ was computed in at most τ time units.*

The terminology 'universal' referring to the regular asynchronous systems means maximal in the sense of the inclusion from Definition 18.

Theorem 20 *a) If $f \subset \Xi_\Phi$, then the function $\pi : \Delta \to P^*(P_n)$ exists,*

$$\Delta = \{(\mu, u) | u \in U, \mu \in \phi_0(u)\} \tag{3}$$

such that $\forall u \in U$,

$$f(u) = \{\Phi^\rho(\mu, u, \cdot) | \mu \in \phi_0(u), \rho \in \pi(\mu, u)\}. \tag{4}$$

b) If π exists such that (3), (4) hold, then $f \subset \Xi_\Phi$.

Proof. a) Let $u \in U$ be arbitrary. From the fact that $f \subset \Xi_\Phi$ and $f(u) \neq \emptyset$ we infer that $\forall \mu \in \phi_0(u)$, the set $\{\rho | \rho \in P_n, \Phi^\rho(\mu, u, \cdot) \in f(u)\}$ is non-empty. We define Δ by equation (3) and $\pi : \Delta \to P^*(P_n)$ by $\forall (\mu, u) \in \Delta$,

$$\pi(\mu, u) = \{\rho | \rho \in P_n, \Phi^\rho(\mu, u, \cdot) \in f(u)\}.$$

(4) is fulfilled.

b) Obvious. ■

Definition 21 *For the regular system $f \subset \Xi_\Phi$, the function π previously defined is called the* ***computation function*** *of f.*

4 Parallel connection

Definition 22 *Consider the systems $f' : U' \to P^*(S^{(n')}), f'' : U'' \to P^*(S^{(n'')})$, $U', U'' \in P^*(S^{(m)})$ with $U' \cap U'' \neq \emptyset$. The system $f' || f'' : U' \cap U'' \to P^*(S^{(n'+n'')})$ defined by*

$$\forall u \in U' \cap U'', (f' || f'')(u) = f'(u) \times f''(u)$$

is called the ***parallel connection*** *of the systems f' and f''.*

Remark 23 *The parallel connection of two systems f' and f'' is the system that represents f', f'' acting independently on each other under the same input $u \in U' \cap U''$.*

5 The parallel connection of the regular systems

Notation 24 *Let be* $\Phi' : \mathbf{B}^{n'} \times \mathbf{B}^m \to \mathbf{B}^{n'}$ *and* $\Phi'' : \mathbf{B}^{n''} \times \mathbf{B}^m \to \mathbf{B}^{n''}$, *for which we denote by* $\Phi'||\Phi'' : \mathbf{B}^{n'+n''} \times \mathbf{B}^m \to \mathbf{B}^{n'+n''}$ *the function* $\forall((\mu', \mu''), \lambda) \in \mathbf{B}^{n'+n''} \times \mathbf{B}^m$,

$$(\Phi'||\Phi'')((\mu', \mu''), \lambda) = (\Phi'(\mu', \lambda), \Phi''(\mu'', \lambda)).$$

Definition 25 *The function* $\Phi : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$ *is given. The (**Boolean partial) derivative of*** Φ_i ***relative to*** μ_j, $i, j \in \{1, ..., n\}$ *is the function* $\dfrac{\partial \Phi_i}{\partial \mu_j} : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}$ *given by* $\forall(\mu, \lambda) \in \mathbf{B}^n \times \mathbf{B}^m$,

$$\frac{\partial \Phi_i}{\partial \mu_j}(\mu, \lambda) = \Phi_i(\mu_1, ..., \mu_j, ..., \mu_n, \lambda) \oplus \Phi_i(\mu_1, ..., \overline{\mu_j}, ..., \mu_n, \lambda).$$

Theorem 26 *We consider the functions* Φ' *and* Φ''. *The following statements are true:*

i) $\forall(\mu, \lambda) \in \mathbf{B}^{n'+n''} \times \mathbf{B}^m, \forall i \in \{1, ..., n'\}, \forall j \in \{n'+1, ..., n'+n''\}$,

$$(\Phi'||\Phi'')_i(\mu_1, ..., \mu_{n'}, ..., \mu_j, ..., \mu_{n'+n''}, \lambda) =$$

$$= (\Phi'||\Phi'')_i(\mu_1, ..., \mu_{n'}, ..., \overline{\mu_j}, ..., \mu_{n'+n''}, \lambda),$$

$\forall(\mu, \lambda) \in \mathbf{B}^{n'+n''} \times \mathbf{B}^m, \forall i \in \{n'+1, ..., n'+n''\}, \forall j \in \{1, ..., n'\}$,

$$(\Phi'||\Phi'')_i(\mu_1, ..., \mu_j, ..., \mu_{n'+1}, ..., \mu_{n'+n''}, \lambda) =$$

$$= (\Phi'||\Phi'')_i(\mu_1, ..., \overline{\mu_j}, ..., \mu_{n'+1}, ..., \mu_{n'+n''}, \lambda);$$

ii) $\forall(\mu, \lambda) \in \mathbf{B}^{n'+n''} \times \mathbf{B}^m$,

$$\forall i \in \{1, ..., n'\}, \forall j \in \{n'+1, ..., n'+n''\}, \frac{\partial(\Phi'||\Phi'')_i}{\partial \mu_j}(\mu, \lambda) = 0,$$

$$\forall i \in \{n'+1, ..., n'+n''\}, \forall j \in \{1, ..., n'\}, \frac{\partial(\Phi'||\Phi'')_i}{\partial \mu_j}(\mu, \lambda) = 0;$$

iii) $\forall(\mu, \lambda) \in \mathbf{B}^{n'+n''} \times \mathbf{B}^m$,

$$(\Phi'||\Phi'')_1(\mu, \lambda) = \Phi'_1(\mu_1, ..., \mu_{n'}, \lambda), \tag{5}$$

$$...$$

$$(\Phi'||\Phi'')_{n'}(\mu, \lambda) = \Phi'_{n'}(\mu_1, ..., \mu_{n'}, \lambda), \tag{6}$$

$$(\Phi'||\Phi'')_{n'+1}(\mu, \lambda) = \Phi''_1(\mu_{n'+1}, ..., \mu_{n'+n''}, \lambda), \tag{7}$$

$$...$$

$$(\Phi'||\Phi'')_{n'+n''}(\mu, \lambda) = \Phi''_{n''}(\mu_{n'+1}, ..., \mu_{n'+n''}, \lambda). \tag{8}$$

Proof. $\forall(\mu,\lambda)\in\mathbf{B}^{n'+n''}\times\mathbf{B}^m,\forall i\in\{1,...,n'\},\forall j\in\{n'+1,...,n'+n''\}$,

$$(\Phi'||\Phi'')_i(\mu_1,...,\mu_{n'},...,\mu_j,...,\mu_{n'+n''},\lambda)=\Phi'_i(\mu_1,...,\mu_{n'},\lambda)=$$

$$=(\Phi'||\Phi'')_i(\mu_1,...,\mu_{n'},...,\overline{\mu_j},...,\mu_{n'+n''},\lambda)$$

i.e. the first part of i) holds true or, equivalently:

$$\frac{\partial(\Phi'||\Phi'')_i}{\partial\mu_j}(\mu,\lambda)=\Phi'_i(\mu_1,...,\mu_{n'},\lambda)\oplus\Phi'_i(\mu_1,...,\mu_{n'},\lambda)=0$$

i.e. the first part of ii) holds true. When i runs in $\{1,...,n'\}$, (5),...,(6) are true themselves. ∎

Lemma 27 *For any $\rho'\in P_{n'},\rho''\in P_{n''}$, the function $\rho'\times\rho'':\mathbf{R}\to\mathbf{B}^{n'+n''}$ is progressive and it belongs to $P_{n'+n''}$.*

Proof. We presume (without loosing the generality) that

$$\rho'(t)=\alpha'^0\cdot\chi_{\{t_0\}}(t)\oplus...\oplus\alpha'^k\cdot\chi_{\{t_k\}}(t)\oplus... \tag{9}$$

$$\rho''(t)=\alpha''^0\cdot\chi_{\{t_0\}}(t)\oplus...\oplus\alpha''^k\cdot\chi_{\{t_k\}}(t)\oplus... \tag{10}$$

with $\alpha'\in\Pi_{n'},\alpha''\in\Pi_{n''}$ and $(t_k)\in Seq$. We infer

$$(\rho'\times\rho'')(t)=(\rho'(t),\rho''(t))=(\alpha'^0,\alpha''^0)\cdot\chi_{\{t_0\}}(t)\oplus...\oplus(\alpha'^k,\alpha''^k)\cdot\chi_{\{t_k\}}(t)\oplus...$$

The function $\rho'\times\rho''$ is progressive because $\forall i\in\{1,...,n'\}$, the set

$$\{k|k\in\mathbf{N},(\alpha'^k,\alpha''^k)_i=1\}=\{k|k\in\mathbf{N},\alpha_i'^k=1\}$$

is infinite and $\forall i\in\{n'+1,...,n'+n''\}$, the set

$$\{k|k\in\mathbf{N},(\alpha'^k,\alpha''^k)_i=1\}=\{k|k\in\mathbf{N},\alpha_i''^k=1\}$$

is infinite too. ∎

Theorem 28 *The functions $\Phi':\mathbf{B}^{n'}\times\mathbf{B}^m\to\mathbf{B}^{n'}$, $\Phi'':\mathbf{B}^{n''}\times\mathbf{B}^m\to\mathbf{B}^{n''}$ are given. For any $\rho'\in P_{n'}$, $\rho''\in P_{n''}$, $\mu'\in\mathbf{B}^{n'}$, $\mu''\in\mathbf{B}^{n''}$, $u\in S^{(m)}$ and any $t\in\mathbf{R}$ we have*

$$(\Phi'||\Phi'')^{\rho'\times\rho''}((\mu',\mu''),u,t)=(\Phi'^{\rho'}(\mu',u,t),\Phi''^{\rho''}(\mu'',u,t)).$$

Proof. We suppose that ρ', ρ'' are like at (9), (10) and Lemma 27 shows that $\rho' \times \rho''$ is a progressive function from $P_{n'+n''}$, thus the statement of the Theorem makes sense.

For $\rho' \in P_{n'}$, $\rho'' \in P_{n''}$, $\mu' \in \mathbf{B}^{n'}$, $\mu'' \in \mathbf{B}^{n''}$, $u \in S^{(m)}$ and $t \in \mathbf{R}$ we can write

$$(\Phi'||\Phi'')^{\rho' \times \rho''}((\mu', \mu''), u, t) = \omega_{-1} \cdot \chi_{(-\infty, t_0)}(t)$$

$$\oplus \omega_0 \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus \omega_k \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

where $\forall k \in \mathbf{N} \cup \{-1\}, \omega_k \in \mathbf{B}^{n'+n''}$ and

$$\omega_{-1} = (\mu', \mu''),$$

$$\omega_{k+1} = (\Phi'||\Phi'')^{(\alpha'^{k+1}, \alpha''^{k+1})}(\omega_k, u(t_{k+1})).$$

We denote by $\omega'_k \in \mathbf{B}^{n'}, \omega''_k \in \mathbf{B}^{n''}$ the first n' coordinates and the last n'' coordinates of ω_k and we remark that $\forall k \in \mathbf{N} \cup \{-1\}$,

$$(\omega'_{-1}, \omega''_{-1}) = (\mu', \mu''),$$

$$(\omega'_{k+1}, \omega''_{k+1}) = (\Phi'||\Phi'')^{(\alpha'^{k+1}, \alpha''^{k+1})}((\omega'_k, \omega''_k), u(t_{k+1})) =$$

$$= (\Phi'^{\alpha'^{k+1}}||\Phi''^{\alpha''^{k+1}})((\omega'_k, \omega''_k), u(t_{k+1})) =$$

$$= (\Phi'^{\alpha'^{k+1}}(\omega'_k, u(t_{k+1})), \Phi''^{\alpha''^{k+1}}(\omega''_k, u(t_{k+1})))$$

thus $(\omega'_k), (\omega''_k)$ fulfill

$$\Phi'^{\rho'}(\mu', u, t) = \omega'_{-1} \cdot \chi_{(-\infty, t_0)}(t) \oplus \omega'_0 \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus \omega'_k \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

$$\Phi''^{\rho''}(\mu'', u, t) = \omega''_{-1} \cdot \chi_{(-\infty, t_0)}(t) \oplus \omega''_0 \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus \omega''_k \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

The conclusion is that

$$(\Phi'||\Phi'')^{\rho' \times \rho''}((\mu', \mu''), u, t) =$$

$$= (\omega'_{-1}, \omega''_{-1}) \cdot \chi_{(-\infty, t_0)}(t) \oplus (\omega'_0, \omega''_0) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus (\omega'_k, \omega''_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ... =$$

$$= (\Phi'^{\rho'}(\mu', u, t), \Phi''^{\rho''}(\mu'', u, t)).$$

■

Notation 29 *We suppose that the systems* $f' : U' \to P^*(S^{(n')}), f'' : U'' \to P^*(S^{(n'')}), U', U'' \in P^*(S^{(m)})$ *are regular i.e.* $f' \subset \Xi_{\Phi'}, f'' \subset \Xi_{\Phi''}$. *Let* $\phi'_0 : U' \to P^*(\mathbf{B}^{n'})$, $\phi''_0 : U'' \to P^*(\mathbf{B}^{n''})$ *be their initial state functions and* $\pi' : \Delta' \to P^*(P_{n'})$, $\pi'' : \Delta'' \to P^*(P_{n''})$ *be their computation functions,*

$$\Delta' = \{(\mu', u') | u' \in U', \mu' \in \phi'_0(u')\},$$

$$\Delta'' = \{(\mu'', u'')|u'' \in U'', \mu'' \in \phi_0''(u'')\}.$$

If $U' \cap U'' \neq \emptyset$, then we use the notations $\phi_{||} : U' \cap U'' \to P^*(\mathbf{B}^{n'+n''})$, $\pi_{||} : \Delta'||\Delta'' \to P^*(P_{n'+n''})$ for the functions $\forall u \in U' \cap U''$,

$$\phi_{||}(u) = \phi_0'(u) \times \phi_0''(u)$$

and respectively

$$\Delta'||\Delta'' = \{((\mu', \mu''), u)|u \in U' \cap U'', \mu' \in \phi_0'(u), \mu'' \in \phi_0''(u)\},$$

$\forall((\mu', \mu''), u) \in \Delta'||\Delta''$,

$$\pi_{||}((\mu', \mu''), u) = \pi'(\mu', u) \times \pi''(\mu'', u).$$

Theorem 30 *If $f' \subset \Xi_{\Phi'}$, $f'' \subset \Xi_{\Phi''}$ and $U' \cap U'' \neq \emptyset$, then $f'||f'' \subset \Xi_{\Phi'||\Phi''}$, its initial state function is $\phi_{||}$ and its computation function is $\pi_{||}$.*

Proof. We prove first that the initial state function of $f' \times f''$ is $\phi_{||}$: $\forall u \in U' \cap U''$,

$$\{z(-\infty + 0)|z \in (f'||f'')(u)\} =$$
$$= \{(x'(-\infty + 0), x''(-\infty + 0))|(x', x'') \in f'(u) \times f''(u)\} =$$
$$= \{(x'(-\infty + 0), x''(-\infty + 0))|x' \in f'(u), x'' \in f''(u)\} =$$
$$= \{x'(-\infty + 0)|x' \in f'(u)\} \times \{x''(-\infty + 0)|x'' \in f''(u)\} =$$
$$= \phi_0'(u) \times \phi_0''(u) = \phi_{||}(u).$$

We infer $\forall u \in U' \cap U''$,

$$(f'||f'')(u) = f'(u) \times f''(u) =$$
$$= \{\Phi'^{\rho'}(\mu', u, \cdot)|\mu' \in \phi_0'(u), \rho' \in \pi'(\mu', u)\} \times$$
$$\times \{\Phi''^{\rho''}(\mu'', u, \cdot)|\mu'' \in \phi_0''(u), \rho'' \in \pi''(\mu'', u)\} =$$
$$= \{(\Phi'^{\rho'}(\mu', u, \cdot), \Phi''^{\rho''}(\mu'', u, \cdot))|$$
$$\mu' \in \phi_0'(u), \mu'' \in \phi_0''(u), \rho' \in \pi'(\mu', u), \rho'' \in \pi''(\mu'', u)\} =$$
$$= \{(\Phi'^{\rho'}(\mu', u, \cdot), \Phi''^{\rho''}(\mu'', u, \cdot))|$$
$$(\mu', \mu'') \in \phi_0'(u) \times \phi_0''(u), \rho' \times \rho'' \in \pi'(\mu', u) \times \pi''(\mu'', u)\} =$$
$$= \{(\Phi'^{\rho'}(\mu', u, \cdot), \Phi''^{\rho''}(\mu'', u, \cdot))|$$
$$(\mu', \mu'') \in \phi_{||}(u), \rho' \times \rho'' \in \pi_{||}((\mu', \mu''), u)\} =$$
$$\overset{\text{Theorem 28}}{=} \{(\Phi'||\Phi'')^{\rho' \times \rho''}((\mu', \mu''), u, \cdot)|$$
$$(\mu', \mu'') \in \phi_{||}(u), \rho' \times \rho'' \in \pi_{||}((\mu', \mu''), u)\}.$$

We apply Theorem 20 b). ■

6 The decomposition of the systems as parallel connection of systems

Theorem 31 *The function* Φ *and the numbers* $n', n'' > 0, n' + n'' = n$ *are given. The following statements are equivalent (see Theorem 26):*

i) $\forall(\mu, \lambda) \in \mathbf{B}^n \times \mathbf{B}^m, \forall i \in \{1, ..., n'\}, \forall j \in \{n' + 1, ..., n\}$,

$$\Phi_i(\mu_1, ..., \mu_{n'}, ..., \mu_j, ..., \mu_n, \lambda) = \Phi_i(\mu_1, ..., \mu_{n'}, ..., \overline{\mu_j}, ..., \mu_n, \lambda),$$

$\forall(\mu, \lambda) \in \mathbf{B}^n \times \mathbf{B}^m, \forall i \in \{n' + 1, ..., n\}, \forall j \in \{1, ..., n'\}$,

$$\Phi_i(\mu_1, ..., \mu_j, ..., \mu_{n'+1}, ..., \mu_n, \lambda) = \Phi_i(\mu_1, ..., \overline{\mu_j}, ..., \mu_{n'+1}, ..., \mu_n, \lambda);$$

ii) $\forall(\mu, \lambda) \in \mathbf{B}^n \times \mathbf{B}^m$,

$$\forall i \in \{1, ..., n'\}, \forall j \in \{n' + 1, ..., n\}, \frac{\partial \Phi_i}{\partial \mu_j}(\mu, \lambda) = 0,$$

$$\forall i \in \{n' + 1, ..., n\}, \forall j \in \{1, ..., n'\}, \frac{\partial \Phi_i}{\partial \mu_j}(\mu, \lambda) = 0;$$

iii) the functions $\Phi' : \mathbf{B}^{n'} \times \mathbf{B}^m \to \mathbf{B}^{n'}$, $\Phi'' : \mathbf{B}^{n''} \times \mathbf{B}^m \to \mathbf{B}^{n''}$ *exist such that* $\forall(\mu, \lambda) \in \mathbf{B}^n \times \mathbf{B}^m$,

$$\Phi_1(\mu, \lambda) = \Phi'_1(\mu_1, ..., \mu_{n'}, \lambda), \tag{11}$$

$$...$$

$$\Phi_{n'}(\mu, \lambda) = \Phi'_{n'}(\mu_1, ..., \mu_{n'}, \lambda), \tag{12}$$

$$\Phi_{n'+1}(\mu, \lambda) = \Phi''_1(\mu_{n'+1}, ..., \mu_{n'+n''}, \lambda), \tag{13}$$

$$...$$

$$\Phi_{n'+n''}(\mu, \lambda) = \Phi''_{n''}(\mu_{n'+1}, ..., \mu_{n'+n''}, \lambda). \tag{14}$$

Proof. i)$\Longleftrightarrow$ii) is obvious.

i)$\Longrightarrow$iii) From the fact that $\forall(\mu, \lambda) \in \mathbf{B}^n \times \mathbf{B}^m$, $\forall i \in \{1, ..., n'\}$, we have

$$\Phi_i(\mu_1, ..., \mu_{n'}, 0, ..., 0, 0) = \Phi_i(\mu_1, ..., \mu_{n'}, 0, ..., 0, 1) =$$
$$= \Phi_i(\mu_1, ..., \mu_{n'}, 0, ..., 1, 0) = \Phi_i(\mu_1, ..., \mu_{n'}, 0, ..., 1, 1) = ...$$
$$... = \Phi_i(\mu_1, ..., \mu_{n'}, 1, ..., 1, 1),$$

we infer the existence of Φ' such that (11),...,(12) are true.

iii)$\Longrightarrow$i) From the existence of Φ' such that $\forall(\mu, \lambda) \in \mathbf{B}^n \times \mathbf{B}^m$, $\forall i \in \{1, ..., n'\}$, $\forall j \in \{n' + 1, ..., n\}$,

$$\Phi_i(\mu_1, ..., \mu_{n'}, ..., \mu_j, ..., \mu_n, \lambda) = \Phi'_i(\mu_1, ..., \mu_{n'}, \lambda) =$$
$$= \Phi_i(\mu_1, ..., \mu_{n'}, ..., \overline{\mu_j}, ..., \mu_n, \lambda),$$

we get that the first part of i) is fulfilled. ∎

Definition 32 *If one of the previous properties i), ii), iii) is true, we say that* ***the coordinates*** $\Phi_1, ..., \Phi_{n'}$ ***do not depend on*** $\mu_{n'+1}, ..., \mu_n$ *and that* ***the coordinates*** $\Phi_{n'+1}, ..., \Phi_n$ ***do not depend on*** $\mu_1, ..., \mu_{n'}$. *The coordinates* $\{1, ..., n'\}$ *and* $\{n'+1, ..., n\}$ *are called* ***separated****. We also say that* $\{1, ..., n\}$ *accepts the* $\Phi-$***partition*** $\{1, ..., n'\}$, $\{n'+1, ..., n\}$.

Theorem 33 *We suppose that* $\Phi_1, ..., \Phi_{n'}$ *do not depend on* $\mu_{n'+1}, ..., \mu_n$ *and that* $\Phi_{n'+1}, ..., \Phi_n$ *do not depend on* $\mu_1, ..., \mu_{n'}$. *Then the functions* $\Phi' : \mathbf{B}^{n'} \times \mathbf{B}^m \to \mathbf{B}^{n'}$, $\Phi'' : \mathbf{B}^{n''} \times \mathbf{B}^m \to \mathbf{B}^{n''}$ *exist such that* $\Phi = \Phi' || \Phi''$.

Proof. From Theorem 31 iii) we have the existence of Φ', Φ'' such that (11),..., (12), (13),..., (14) are fulfilled. We denote by μ', μ'' the first n' coordinates of $\mu \in \mathbf{B}^n$ and respectively the last n'' coordinates of μ. We have $\forall (\mu, \lambda) \in \mathbf{B}^n \times \mathbf{B}^m$,

$$\Phi((\mu', \mu''), \lambda) = (\Phi_1((\mu', \mu''), \lambda), ...$$
$$..., \Phi_{n'}((\mu', \mu''), \lambda), \Phi_{n'+1}((\mu', \mu''), \lambda), ..., \Phi_n((\mu', \mu''), \lambda)) =$$
$$= (\Phi'_1(\mu', \lambda), ..., \Phi'_{n'}(\mu', \lambda), \Phi''_1(\mu'', \lambda), ..., \Phi''_{n''}(\mu'', \lambda)) =$$
$$= (\Phi'(\mu', \lambda), \Phi''(\mu'', \lambda)) = (\Phi' || \Phi'')((\mu', \mu''), \lambda).$$

■

Notation 34 *Let be the system* $f \subset \Xi_\Phi$, *where* $\Phi : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$, $f : U \to P^*(S^{(n)})$ *and* $\forall u \in U$,

$$f(u) = \{\Phi^\rho(\mu, u, \cdot) | \mu \in \phi_0, \rho \in \pi(\mu, u)\}.$$

We suppose that $n', n'' > 0, n' + n'' = n$ *are given. Then we denote by* $\phi'_0 : U \to P^*(\mathbf{B}^{n'})$, $\pi' : \Delta' \to P^*(P_{n'})$ *the functions* $\forall u \in U$,

$$\phi'_0(u) = \{(\mu_1, ..., \mu_{n'}) | \mu \in \phi_0(u)\},$$
$$\Delta' = \{(\mu', u) | u \in U, \mu' \in \phi'_0(u)\}$$

and respectively $\forall (\mu', u) \in \Delta'$,

$$\pi'(\mu', u) = \{(\rho_1, ..., \rho_{n'}) | \exists \mu'' \in \mathbf{B}^{n''}, (\mu', \mu'') \in \phi_0(u), \rho \in \pi((\mu', \mu''), u)\}.$$

The functions $\phi''_0 : U \to P^*(\mathbf{B}^{n''}), \pi'' : \Delta'' \to P^*(P_{n''})$ *are obviously defined in this moment.*

Notation 35 *If* $\forall u \in U$, $\forall \mu \in \phi_0(u)$, $\forall \rho' \in \pi'(\mu', u)$, $\forall \rho'' \in \pi''(\mu'', u)$, $\exists \widetilde{\rho} \in \pi(\mu, u)$, $\forall t \in \mathbf{R}$,

$$\Phi^{\widetilde{\rho}}(\mu, u, t) = \Phi^{\rho' \times \rho''}(\mu, u, t),$$

then we denote $\forall u \in U, \forall \mu \in \phi_0(u), \pi(\mu, u) \approx \pi'(\mu', u) \times \pi''(\mu'', u)$. π', π'' *are the previous ones,* μ', ρ' *are the first* n' *coordinates of* μ, ρ *and* μ'', ρ'' *are the last* n'' *coordinates of* μ, ρ, *where* $n' + n'' = n$.

Remark 36 *We can see that* $\forall u \in U, \phi_0(u) \subset \phi_0'(u) \times \phi_0''(u)$ *and* $\forall u \in U, \forall \mu \in \phi_0(u), \pi(\mu, u) \subset \pi'(\mu', u) \times \pi''(\mu'', u)$ *hold.*

Theorem 37 *The regular system* $f \subset \Xi_\Phi$ *is given and we suppose that the functions* $\Phi' : \mathbf{B}^{n'} \times \mathbf{B}^m \to \mathbf{B}^{n'}$, $\Phi'' : \mathbf{B}^{n''} \times \mathbf{B}^m \to \mathbf{B}^{n''}$ *exist such that* $n', n'' > 0, n' + n'' = n$ *and* $\forall \mu \in \mathbf{B}^n, \forall \lambda \in \mathbf{B}^m$, *the equations (11),..., (12), (13),..., (14) are fulfilled. Then* $\Phi = \Phi' || \Phi''$ *and the systems* $f' \subset \Xi_{\Phi'}, f'' \subset \Xi_{\Phi''}$ *defined by* $f' : U \to P^*(S^{(n')})$, $f'' : U \to P^*(S^{(n'')}), \forall u \in U$,

$$f'(u) = \{\Phi'^{\rho'}(\mu', u, \cdot) | \mu' \in \phi_0'(u), \rho' \in \pi'(\mu', u)\},$$

$$f''(u) = \{\Phi''^{\rho''}(\mu'', u, \cdot) | \mu'' \in \phi_0''(u), \rho'' \in \pi''(\mu'', u)\}$$

satisfy $f \subset f' || f''$. *If* $\forall u \in U, \phi_0(u) = \phi_0'(u) \times \phi_0''(u)$ *and* $\forall u \in U, \forall \mu \in \phi_0(u), \pi(\mu, u) \approx \pi'(\mu', u) \times \pi''(\mu'', u)$, *then we have* $f = f' || f''$.

Proof. The fact that $\Phi = \Phi' || \Phi''$ results from Theorem 33. We denote like previously with μ', μ'' the first n' coordinates of $\mu \in \mathbf{B}^n$ and the last n'' coordinates of μ and the notations are similar for ρ', ρ'' and $\rho \in P_n$. We have $\forall u \in U$,

$$f(u) = \{\Phi^\rho(\mu, u, \cdot) | \mu \in \phi_0(u), \rho \in \pi(\mu, u)\} =$$

$$= \{(\Phi' || \Phi'')^{\rho' \times \rho''}((\mu', \mu''), u, \cdot) | \mu \in \phi_0(u), \rho \in \pi(\mu, u)\} =$$

$$\overset{\text{Theorem 28}}{=} \{(\Phi'^{\rho'}(\mu', u, \cdot), \Phi''^{\rho''}(\mu'', u, \cdot)) | \mu \in \phi_0(u), \rho \in \pi(\mu, u)\} \subset$$

$$\subset \{(\Phi'^{\rho'}(\mu', u, \cdot), \Phi''^{\rho''}(\mu'', u, \cdot))|$$

$$|(\mu', \mu'') \in \phi_0'(u) \times \phi_0''(u), \rho' \times \rho'' \in \pi'(\mu', u) \times \pi''(\mu'', u)\} =$$

$$= \{\Phi'^{\rho'}(\mu', u, \cdot) | \mu' \in \phi_0'(u), \rho' \in \pi'(\mu', u)\} \times$$

$$\times \{\Phi''^{\rho''}(\mu'', u, \cdot) | \mu'' \in \phi_0''(u), \rho'' \in \pi''(\mu'', u)\} =$$

$$= f'(u) \times f''(u) = (f' || f'')(u).$$

The second statement of the theorem is obvious. ■

References

[1] Serban E. Vlad, Teoria sistemelor asincrone, Editura Pamantul, Pitesti, 2007.