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$\boldsymbol{R} \to \boldsymbol{B}_2$ DIFFERENTIABLE FUNCTIONS

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Abstract *The $\boldsymbol{R} \to \boldsymbol{B}_2$ functions may be interpreted to represent*

- *subsets of $\boldsymbol{R}$*
- *models of the electrical signals*
- *propositions having a logical value that depends on time*

Our purpose is to define and characterize the $\boldsymbol{R} \to \boldsymbol{B}_2$ differentiable functions.

1. Introduction. Basic Definitions

1.1 The $\boldsymbol{R} \to \boldsymbol{B}_2$ differentiable functions are important for at least two reasons:

- they model the behavior of the inertial digital devices and the study of the models, called asynchronous automata, has been a field of interest for us.

- they have strong resemblance with the real differentiable functions, making possible analogies. We shall give comparisons between theorems characterizing these functions and the corresponding theorems referring to real functions.

It is interesting to leave the study open, making use of the ideas presented here:

- of the subsets of $\boldsymbol{R}$ (or of any ordered set)
- of a suitable temporal logic

1.2 $\boldsymbol{B}_2 = \{0,1\}$ is endowed with the laws:

$$\begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \qquad \begin{array}{c|cc} \cup & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}$$

table 1.2

and it is a field relative to $\oplus, \cdot$. It has the discrete topology and the order $0 \le 1$.

1.3 A sequence $a : \boldsymbol{N} \to \boldsymbol{B}_2, a(n) = a_n, n \in \boldsymbol{N}$ *converges to the limit* $\tilde{a} \in \boldsymbol{B}_2$ if

$$\exists N \in \boldsymbol{N}, \forall n > N, a_n = \tilde{a}$$

For example, the increasing sequences and the decreasing sequences are convergent.

1.4 If $(a_n), (b_n)$ are convergent, then $(a_n \oplus b_n), (a_n \cdot b_n)$ are convergent and

$$\lim_{n\to\infty} (a_n \oplus b_n) = \lim_{n\to\infty} a_n \oplus \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n\cdot b_n)=(\lim_{n\to\infty}a_n)\cdot(\lim_{n\to\infty}b_n)$$

1.5 Let $x:\boldsymbol{R}\to\boldsymbol{B}_2$. There are defined for $A\subset\boldsymbol{R}$:

$$\bigcup_{\xi\in A}x(\xi)=\begin{cases}1, if\ \exists\xi\in A, x(\xi)=1\\0, if\ \forall\xi\in A, x(\xi)=0\end{cases},\ \bigcup_{\xi\in\emptyset}x(\xi)=0$$

$$\bigcap_{\xi\in A}x(\xi)=\begin{cases}0, if\ \exists\xi\in A, x(\xi)=0\\1, if\ \forall\xi\in A, x(\xi)=1\end{cases},\ \bigcap_{\xi\in\emptyset}x(\xi)=1$$

We have for $B\subset\boldsymbol{R}$:

$$A\subset B\Rightarrow\bigcup_{\xi\in A}x(\xi)\le\bigcup_{\xi\in B}x(\xi)\ and\ \bigcap_{\xi\in A}x(\xi)\ge\bigcap_{\xi\in B}x(\xi)$$

1.6 The *support of* x is the set:

$$supp\ x=\{t\mid t\in\boldsymbol{R}, x(t)=1\}$$

We note with $\chi_A:\boldsymbol{R}\to\boldsymbol{B}_2$ the *characteristic function* of the set A. It is true:

$$x(t)=\chi_{supp\ x}(t), t\in\boldsymbol{R}$$

1.7 We have the next properties:

$$(x\oplus y)(t)=x(t)\oplus y(t)=\chi_{supp\ x}(t)\oplus\chi_{supp\ y}(t)=\chi_{supp\ x\,\Delta\, supp\ y}(t)$$

$$(x\cdot y)(t)=x(t)\cdot y(t)=\chi_{supp\ x}(t)\cdot\chi_{supp\ y}(t)=\chi_{supp\ x\,\wedge\, supp\ y}(t)$$

$$A\subset B\Leftrightarrow\chi_A(t)\le\chi_B(t)$$

where $x,y:\boldsymbol{R}\to\boldsymbol{B}_2$, $t\in\boldsymbol{R}$ and $A,B\subset\boldsymbol{R}$.

$\boldsymbol{B}_2^{\boldsymbol{R}}$ and $\{A\mid A\subset\boldsymbol{R}\}$ are isomorphic commutative unitary rings.

1.8 $\emptyset$ is a finite set having 0 elements, where 0 is considered to be an even natural number.

$G_f=\{A\mid A\subset\boldsymbol{R}, A\ is\ finite\}$ is an abelian group relative to Δ and $\emptyset$ is the neuter element.

1.9 It is defined for $x:\boldsymbol{R}\to\boldsymbol{B}_2$ and $A\subset\boldsymbol{R}$ so that $A\wedge supp\ x\in G_f$ - condition of convergence of a generalized series - the modulo 2 summation

$$\mathop{\Xi}_{\xi\in A}x(\xi)=\begin{cases}1, if\ |A\wedge supp\ x|\ is\ odd\\0, if\ |A\wedge supp\ x|\ is\ even\end{cases}$$

resulting

$$\mathop{\Xi}_{\xi\in\emptyset}x(\xi)=0$$

2. The Limit

2.1 Let $x:\boldsymbol{R}\to\boldsymbol{B}_2$ and $\varepsilon_n\in\boldsymbol{R}, n\in\boldsymbol{N}$ a real positive sequence that is strictly decreasing convergent to 0. For any $t\in\boldsymbol{R}$, the sequence $\bigcup_{\xi\in(t-\varepsilon_n,t)}x(\xi)$, $n\in\boldsymbol{N}$ is decreasing so that it is convergent. Its limit - which does not depend on (ε_n) - is called the *left superior limit* of x in t and it is noted with $\overline{x(t-0)}$, $\lim\limits_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \bigcup\limits_{\xi\in(t-\varepsilon,t)} x(\xi)$ or $\limsup\limits_{\xi\to t^-} x(\xi)$. We have:

$$\overline{x(t-0)} = \begin{cases} 1, \text{ if } \forall\varepsilon>0, \exists\xi\in(t-\varepsilon,t), x(\xi)=1 \\ 0, \text{ else} \end{cases}$$

2.2 Let us suppose now that (ε'_n) is a real sequence that is strictly increasing convergent to ∞. The sequence $\bigcup\limits_{\xi\in(\varepsilon'_n,\infty)} x(\xi), n\in \mathbf{N}$ is decreasing and convergent. Its limit – that does not depend on (ε'_n) - is called the (*left*) *superior limit of* x in ∞ and it is noted with $\lim\limits_{\varepsilon'\to\infty} \bigcup\limits_{\xi\in(\varepsilon',\infty)} x(\xi)$ or $\limsup\limits_{\xi\to\infty} x(\xi)$. It is true:

$$\limsup_{\xi\to\infty} x(\xi) = \begin{cases} 1, \text{ if } \forall\varepsilon', \exists\xi>\varepsilon', x(\xi)=1 \\ 0, \text{ else} \end{cases}$$

2.3 In a similar manner to 2.1, 2.2 there are also defined:

$$\underline{x(t-0)} = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \bigcap_{\xi\in(t-\varepsilon,t)} x(\xi) = \liminf_{\xi\to t^-} x(\xi);\ \lim_{\varepsilon'\to\infty} \bigcap_{\xi\in(\varepsilon',\infty)} x(\xi) = \liminf_{\xi\to\infty} x(\xi)$$

$$\overline{x(t+0)} = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \bigcup_{\xi\in(t,t+\varepsilon)} x(\xi) = \limsup_{\xi\to t^+} x(\xi);\ \lim_{\varepsilon'\to-\infty} \bigcup_{\xi\in(-\infty,\varepsilon')} x(\xi) = \limsup_{\xi\to-\infty} x(\xi)$$

$$\underline{x(t+0)} = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \bigcap_{\xi\in(t,t+\varepsilon)} x(\xi) = \liminf_{\xi\to t^+} x(\xi);\ \lim_{\varepsilon'\to-\infty} \bigcap_{\xi\in(-\infty,\varepsilon')} x(\xi) = \liminf_{\xi\to-\infty} x(\xi)$$

2.4 **Remark** There are two categories of dual notions in this theory:

- *superior* and *inferior*, because of $\bigcup$, $\bigcap$, respectively $\geq_{\mathbf{B}_2}, \leq_{\mathbf{B}_2}$
- *left* and *right*, because of $>_{\mathbf{R}}, <_{\mathbf{R}}$.

Sometimes, the dual notions will be only mentioned, not defined and/or studied - the way that we have already done at 2.3.

2.5 a) For $x:\mathbf{R}\to\mathbf{B}_2$ and $t\in\mathbf{R}\vee\{\infty\}$, the following statements are equivalent:

- the left superior limit and the left inferior limit of x in t are equal
- $\exists x^0\in\mathbf{B}_2, \exists t'<t, \forall\xi\in(t',t), x(\xi)=x^0$

If one of them is satisfied, we say that x *has a left limit in* t, or that *the left limit of* x *in* t *exists*. This left limit x^0 is noted

- with $x(t-0)$ or $\lim\limits_{\xi\to t^-} x(\xi)$, if $t\in\mathbf{R}$
- with $\lim\limits_{\xi\to\infty} x(\xi)$, if $t=\infty$.

a*) In a similar manner we refer, for $t\in\mathbf{R}\vee\{-\infty\}$, to the *right limit of* x *in* t noted:

- with $x(t+0)$ or $\lim\limits_{\xi \to t^+} x(\xi)$, if $t \in \mathbf{R}$

- with $\lim\limits_{\xi \to -\infty} x(\xi)$, if $t = -\infty$.

2.6 The following situations are also interesting:

- $x(t-0) = x(t+0) \overset{not}{=} \lim\limits_{\xi \to t} x(\xi)$: x has a *limit* in $t \in \mathbf{R}$

- $\overline{x(t-0)} = \overline{x(t+0)} \overset{not}{=} \limsup\limits_{\xi \to t} x(\xi)$: x has a *superior limit* in $t \in \mathbf{R}$

- $\underline{x(t-0)} = \underline{x(t+0)} \overset{not}{=} \liminf\limits_{\xi \to t} x(\xi)$: x has an *inferior limit* in $t \in \mathbf{R}$

2.7 The existence of the left limit of x in t is equivalent to the fact that for any real sequence (ξ_n) that is strictly increasing convergent to t, the sequence $(x(\xi_n))$ is convergent.

The existence of the limit of x in t is equivalent to the situation when for any real sequence (ξ_n) that is convergent to t and $\xi_n \neq t, n \in \mathbf{N}$, the sequence $(x(\xi_n))$ is convergent.

2.8 Let $x : A \to \mathbf{B}_2, A \subset \mathbf{R}$ and we refer to the previous types of limits, under some conditions stated on A:

- the left (superior, inferior) limit in t: $t \in A'$, where

$$A' \overset{def}{=} \{t \mid t \in \mathbf{R} \vee \{\infty\}, \forall t' \in \mathbf{R}, \mathit{if}\ t' < t, \mathit{then}\ A \wedge (t', t) \neq \emptyset\}$$ and we ask that $A' \neq \emptyset$

- the right (superior, inferior) limit in t: $t \in A^*$, where

$$A^* \overset{def}{=} \{t \mid t \in \mathbf{R} \vee \{-\infty\}, \forall t' \in \mathbf{R}, \mathit{if}\ t' > t, \mathit{then}\ A \wedge (t, t') \neq \emptyset\}$$ and we ask that $A^* \neq \emptyset$

- the (superior, inferior) limit in t: $t \in A' \wedge A^*$ and we ask that $A' \wedge A^* \neq \emptyset$.

For example, $\mathbf{R}' = \mathbf{R} \vee \{\infty\}, \mathbf{R}^* = \mathbf{R} \vee \{-\infty\}$.

The point is to let, in these new definitions, ξ run not in subsets of $\mathbf{R}$, but in subsets of A. The definitions of the limits of x relative to subsets of $\mathbf{R}$ are obvious now.

2.9 All the previous definitions refer to binary numbers if t is fixed and to binary valued functions, if t runs in a subset of $\mathbf{R}$.

2.10 We have the following notations:

$$Lim_A(t) = \{x \mid x : A \to \mathbf{B}_2, \mathit{x\ has\ a\ limit\ in\ t}\}, t \in A' \wedge A^*$$
$$\overline{Lim}_A(t) = \{x \mid x : A \to \mathbf{B}_2, \mathit{x\ has\ a\ superior\ limit\ in\ t}\}, t \in A' \wedge A^*$$
$$\underline{Lim}_A(t) = \{x \mid x : A \to \mathbf{B}_2, \mathit{x\ has\ an\ inferior\ limit\ in\ t}\}, t \in A' \wedge A^*$$
$$Lim'_A(t) = \{x \mid x : A \to \mathbf{B}_2, \mathit{x\ has\ a\ left\ limit\ in\ t}\}, t \in A'$$
$$Lim^*_A(t) = \{x \mid x : A \to \mathbf{B}_2, \mathit{x\ has\ a\ right\ limit\ in\ t}\}, t \in A^*$$

If, in the previous notations, the subscript 'A' is missing, then we take $A = \mathbf{R}$ and if 't' is missing, then we refer to all possible t. For example

$$x \in Lim' \Leftrightarrow x \in Lim'_{\mathbf{R}}(t), \forall t \in \mathbf{R} \vee \{\infty\}.$$

2.11 **Examples** a) Let $H \subset \boldsymbol{R}$ be a set with $\forall a < b, (a,b) \wedge H \in G_f$, for example $H = \emptyset, H$ is finite, $H = \boldsymbol{N}$ or $H = \boldsymbol{Z}$. The limit of $c \oplus \chi_H : \boldsymbol{R} \to \boldsymbol{B}_2$, where $c \in \boldsymbol{B}_2$ is the constant function, exists in any $t \in \boldsymbol{R}$ and is equal to c and, as a special case, the limit of c is c. Let us also remark that:

$$\lim_{\xi \to \infty} (c \oplus \chi_H)(\xi) = \begin{cases} c, \textit{if } H \textit{ is superiorly bounded} \\ \textit{does not exist, otherwise} \end{cases}$$

b) $x = \chi_{\{1,1/2,1/3,...\}} : \boldsymbol{R} \to \boldsymbol{B}_2$ has null limit everywhere, except for the origin. In $t = 0$, we have:

$$x(0-0) = 0, \overline{x(0+0)} = 1, \underline{x(0+0)} = 0$$

c) Let x be one of $\chi_{(a,b)}, \chi_{[a,b)}, \chi_{(a,b]}, \chi_{[a,b]} : \boldsymbol{R} \to \boldsymbol{B}_2$. Because

$$x(t-0) = \chi_{(a,b]}(t), t \in \boldsymbol{R} \text{ and } \lim_{t \to \infty} x(t) = 0$$

$$x(t+0) = \chi_{[a,b)}(t), t \in \boldsymbol{R} \text{ and } \lim_{t \to -\infty} x(t) = 0$$

we infer that x has limit everywhere but in $t \in \{a,b\}$.

d) The function of Dirichlet $x : \boldsymbol{R} \to \boldsymbol{B}_2$

$$x(t) = \begin{cases} 1, \textit{if } t \textit{ is rational} \\ 0, \textit{else} \end{cases}$$

has limit (left limit, right limit) in no point $t \in \boldsymbol{R}$ $(t \in \boldsymbol{R} \vee \{\infty\}, t \in \boldsymbol{R} \vee \{-\infty\})$. However:

$$\limsup_{\xi \to t} x(\xi) = 1, t \in \boldsymbol{R} \text{ and } \limsup_{\xi \to \infty} x(\xi) = \limsup_{\xi \to -\infty} x(\xi) = 1$$

$$\liminf_{\xi \to t} x(\xi) = 0, t \in \boldsymbol{R} \text{ and } \liminf_{\xi \to \infty} x(\xi) = \liminf_{\xi \to -\infty} x(\xi) = 0$$

2.12 $Lim_A(t), Lim'_A(t), Lim^*_A(t),$ respectively Lim_A, Lim'_A, Lim^*_A are commutative unitary rings and the limit operators act as binary valued morphisms (see also 1.4).

$\overline{Lim}_A(t), \underline{Lim}_A(t),$ respectively $\overline{Lim}_A, \underline{Lim}_A$ are not rings.

3. The Derivative and the Variation

3.1 The following numbers or functions are defined, depending on the fact if t is fixed or variable:

3.1.1 the *derivative* of $x \in Lim_A(t)$ in t (of $x \in Lim_A$)

$$Dx(t) = \lim_{\xi \to t} x(\xi) \oplus x(t)$$

3.1.2 the *superior derivative* of $x \in \overline{Lim}_A(t)$ in t (of $x \in \overline{Lim}_A$)

$$\overline{D}x(t) = \limsup_{\xi \to t} x(\xi) \oplus x(t)$$

3.1.3 the *left superior derivative* of $x : A \to \boldsymbol{B}_2$ in t (of $x : A \to \boldsymbol{B}_2$)

$$\overline{D}' x(t) = \limsup_{\xi \to t^-} x(\xi) \oplus x(t)$$

3.1.4 the *right superior derivative* of $x : A \to \boldsymbol{B}_2$ in t (of $x : A \to \boldsymbol{B}_2$)

$$\overline{D}^{*} x(t) = \limsup_{\xi \to t^{+}} x(\xi) \oplus x(t)$$

3.1.5 the *inferior derivative* of $x \in \underline{Lim}_A(t)$ in t (of $x \in \underline{Lim}_A$)

$$\underline{D}x(t) = \liminf_{\xi \to t} x(\xi) \oplus x(t)$$

3.1.6 the *left inferior derivative* of $x : A \to \boldsymbol{B}_2$ in t (of $x : A \to \boldsymbol{B}_2$)

$$\underline{D}^{'} x(t) = \liminf_{\xi \to t^{-}} x(\xi) \oplus x(t)$$

3.1.7 the *right inferior derivative* of $x : A \to \boldsymbol{B}_2$ in t (of $x : A \to \boldsymbol{B}_2$)

$$\underline{D}^{*} x(t) = \liminf_{\xi \to t^{+}} x(\xi) \oplus x(t)$$

3.1.8 the *left derivative* of $x \in Lim_A^{'}(t)$ in t (of $x \in Lim_A^{'}$)

$$D^{'} x(t) = \lim_{\xi \to t^{-}} x(\xi) \oplus x(t)$$

3.1.9 the *right derivative* of $x \in Lim_A^{*}(t)$ in t (of $x \in Lim_A^{*}$)

$$D^{*} x(t) = \lim_{\xi \to t^{+}} x(\xi) \oplus x(t)$$

We explicitly point out that $t \in A$ implies the fact that t cannot equal ∞ or $-\infty$ in these definitions.

3.2 Let L be a ring of derivable functions like in 2.12 and let d be a derivation operator. The next equations are true:

$$d(x \oplus y)(t) = dx(t) \oplus dy(t)$$
$$d(x \cdot y)(t) = x(t) \cdot dy(t) \oplus y(t) \cdot dx(t) \oplus dx(t) \cdot dy(t)$$

where $x, y \in L$ and t belongs to a subset of A.

3.3 Let $x \in Lim_A$ and $H \subset A \wedge A' \wedge A^*$. We say that x has a *finite variation* on H if

$$\{t \mid t \in H, Dx(t) = 1\} \in G_f$$

and if so, we define the number

$$V_H x = \underset{\xi \in H}{\Xi} Dx(\xi)$$

called the *variation of x on H*.

In a similar manner there are put the conditions and there are defined the numbers

$$\overline{V}_H x, \overline{V}^{'}_H x, \overline{V}^{*}_H x, \underline{V}_H x, \underline{V}^{'}_H x, \underline{V}^{*}_H x, V^{'}_H x, V^{*}_H x.$$

4. The Continuity

4.1 All the derivatives from 3.1, if equal to 0, define a point (a set $H \subset A$) of continuity for x:

4.1.1 $x \in Lim_A(t)$ is *continuous in t* ($x \in Lim_A$ is *continuous on H*) if

$$Dx(t)=0 \quad (\forall t\in H)$$

The set of the continuous functions in t (of the continuous functions on A) is noted with $Cont_A(t)$ (with $Cont_A$).

...

4.1.9 $x\in Lim^*_A(t)$ is *right continuous in* t ($x\in Lim^*_A$ is *right continuous on* H) if

$$D^*x(t)=0 \quad (\forall t\in H)$$

The set of the right continuous functions in t (of the right continuous functions on A) is noted with $Cont^*_A(t)$ (with $Cont^*_A$).

4.2 If $A=\mathbf{R}$ then, in the previous notations, the subscript $'A'$ is missing.

4.3 We underline, in order to fix the ideas, what means $x\in Cont_A(t)$:

a) $t\in A$ and $\forall\varepsilon>0, A\wedge(t-\varepsilon,t)\neq\emptyset, A\wedge(t,t+\varepsilon)\neq\emptyset$

b) It is satisfied one of the following equivalent conditions:

- $x\in Lim_A(t)$ and $\lim\limits_{\xi\to t} x(\xi)=x(t)$
- $\exists\varepsilon>0,\forall\xi\in A\wedge(t-\varepsilon,t+\varepsilon), x(\xi)=x(t)$
- for any sequence $\xi_n\in A, n\in \mathbf{N}$ convergent to $t\in A$, the sequence $(x(\xi_n))$ converges to $x(t)$.

4.4 The sets $Cont_A(t), Cont'_A(t), Cont^*_A(t),$ as well as $Cont_A, Cont'_A, Cont^*_A$ are commutative unitary rings (see 2.12), while the other sets are not.

5. The Differentiability

5.1 All the derivatives from 3.1 and all the continuities from 4.1 define different concepts of differentiability in a point (on a set $H\subset A$) of $x:A\to \mathbf{B}_2$:

5.1.1 x is *differentiable in* t_0 (x is *differentiable on* H) if $(\forall t_0\in H)$ there exist $a', a^*\in \mathbf{B}_2$ and $\omega\in Cont_A(t_0)$ so that:

$$\lim_{\xi\to t_0}\omega(\xi)=0$$

$$x(t)\oplus x(t_0)=a'\cdot\chi_{(-\infty,t_0)}(t)\oplus a^*\cdot\chi_{(t_0,\infty)}(t)\oplus\omega(t),\ t\in A$$

The set of the differentiable functions in t_0 (of the differentiable functions on A) is noted with $Diff_A(t_0)$ (of $Diff_A$)

...

5.1.7 x is *right inferior differentiable in* t_0 (x is *right inferior differentiable on* H) if $(\forall t_0\in H)$ there exist $a^*\in \mathbf{B}_2$ and $\omega\in \underline{Cont}^*_A(t_0)$ so that

$$\liminf_{\xi\to t_0^+}\omega(\xi)=0$$

$$x(t)\oplus x(t_0)=a^*\cdot\chi_{(t_0,\infty)}(t)\oplus\omega(t),\ t\in A$$

The set of the right inferior differentiable functions in t_0 (of the right inferior differentiable functions on A) is noted with $\underline{Diff}^{\,*}_A(t_0)$ (with $\underline{Diff}^{\,*}_A$)

5.1.8 x is *left differentiable in* t_0 (x is *left differentiable on* H) if $(\forall t_0 \in H)$ there exist $a' \in \boldsymbol{B}_2$ and $\omega \in Cont^{'}_A(t_0)$ so that

$$\lim_{\xi \to t_0^-} \omega(\xi) = 0$$

$$x(t) \oplus x(t_0) = a' \cdot \chi_{(-\infty, t_0)}(t) \oplus \omega(t), \ t \in A$$

the corresponding set being noted with $Diff^{'}_A(t_0)$ (with $Diff^{'}_A$, when $H = A$).

...

5.2 Relative to the definitions 5.1.1,…,5.1.9 we make the next -easy to prove- remarks:

$$Diff_A(t_0) = Lim^{'}_A(t_0) \wedge Lim^{*}_A(t_0)$$

$$x \in Diff_A(t_0) \Rightarrow a' = D'x(t_0), a^* = D^*x(t_0) \quad \text{(see 5.1.1)}$$

$$Diff_A(t_0) = \overline{Diff}_A(t_0)$$

$$Diff^{'}_A(t_0) = Lim^{'}_A(t_0)$$

$$x \in Diff^{'}_A(t_0) \Rightarrow a' = D'x(t_0) \quad \text{(see 5.1.8)}$$

$$Diff^{'}_A(t_0) = \overline{Diff}^{\,'}_A(t_0)$$

$$Diff^{*}_A(t_0) = Lim^{*}_A(t_0)$$

$$x \in Diff^{*}_A(t_0) \Rightarrow a^* = D^*x(t_0)$$

$$Diff^{*}_A(t_0) = \overline{Diff}^{\,*}_A(t_0)$$

The following sets are commutative unitary rings

$$Diff_A(t_0), Diff^{'}_A(t_0), Diff^{*}_A(t_0), Diff_A, Diff^{'}_A, Diff^{*}_A$$

and the following sets are not rings

$$\underline{Diff}_A(t_0), \underline{Diff}^{\,'}_A(t_0), \underline{Diff}^{\,*}_A(t_0), \underline{Diff}_A, \underline{Diff}^{\,'}_A, \underline{Diff}^{\,*}_A$$

5.3 Like before, if $A = \boldsymbol{R}$, then the subscript $'A'$ will be missing.

5.4 **Theorem** (*of representation of the differentiable functions*) For $x : \boldsymbol{R} \to \boldsymbol{B}_2$, the next statements are equivalent:

a) $x \in Diff$

b) there exist the families $\alpha^z, \beta^z \in \boldsymbol{B}_2, t_z \in \boldsymbol{R}, z \in \boldsymbol{Z}$ so that

b.1) $... < t_{-1} < t_0 < t_1 < ...$

b.2) $\forall a < b, (a,b) \wedge \{t_z \mid z \in \boldsymbol{Z}\} \in G_f$

b.3) $x(t) = ... \oplus \alpha^0 \cdot \chi_{(t_0,t_1)}(t) \oplus \beta^1 \cdot \chi_{\{t_1\}}(t) \oplus \alpha^1 \cdot \chi_{(t_1,t_2)}(t) \oplus \beta^2 \cdot \chi_{\{t_2\}}(t) \oplus ...$

The proof of the theorem makes use of the first equation 5.2.

5.5 A family $t_z \in \boldsymbol{R}, z \in \boldsymbol{Z}$ like at 5.4.4 is called *strictly increasing* (condition b.1)) *locally finite* (condition b.2)), shortly SILF.

5.6 **Remarks** a) A SILF real family (t_z) resembles $\boldsymbol{N}$ or $\boldsymbol{Z}$: it is discrete and it gives, when related to asynchronous automata, the discrete time. The differentiable functions $x \in Diff$ model -as we have said at 1.1- the electrical signals of the digital devices: in any bounded time interval (a,b), such a signal can switch (change its value) a finite number of times and its model may have a finite number of points of discontinuity $a = t_0 < t_1 < ... < t_n = b$.

b) The form of reprezentation of the differentiable functions from 5.4 b) is not unique, for example:

$$\chi_{(t_z,t_{z+1})}(t) = \chi_{(t_z,t')}(t) \oplus \chi_{\{t'\}}(t) \oplus \chi_{(t',t_{z+1})}(t)$$

is true, where $t \in \boldsymbol{R}$ and $t_z < t' < t_{z+1}$.

c) The finite sums of $\boldsymbol{R} \to \boldsymbol{B}_2$ functions always make sense, but the countable sums of such functions do not always make sense. The countable summation 5.4 b.3) gives a convergent sum for any $t \in \boldsymbol{R}$ because the supports of the functions $\chi_{(t_z,t_{z+1})}, \chi_{\{t_z\}}, z \in \boldsymbol{Z}$ are disjoint (they form a partition of $\boldsymbol{R}$).

5.7 Let $x \in Diff$ be given by the formula 5.4 b.3), where (t_z) is SILF. The next equations are satisfied for $t \in \boldsymbol{R}$:

$$x(t-0) = ... \oplus \alpha^0 \cdot \chi_{(t_0,t_1]}(t) \oplus \alpha^1 \cdot \chi_{(t_1,t_2]}(t) \oplus ...$$

$$D'x(t) = ... \oplus (\alpha^{-1} \oplus \beta^0) \cdot \chi_{\{t_0\}}(t) \oplus (\alpha^0 \oplus \beta^1) \cdot \chi_{\{t_1\}}(t) \oplus ...$$

5.8 If $x \in Diff$, then $supp\ D'x, supp\ D^*x$ are locally finite, i.e.

$$\forall a < b, (a,b) \wedge supp\ D'x, (a,b) \wedge supp\ D^*x \in G_f$$

5.9 If $x:[a,b] \to \boldsymbol{B}_2$ satisfies $x \in Diff_{[a,b]}(t), t \in (a,b)$, then $x(a+0)$ and $x(b-0)$ exist.

5.10 It is convenient to write $x \in Lim_{[a,b]}$ instead of $x \in Lim_{[a,b]}(t), t \in (a,b)$, $x \in Lim'_{[a,b]}$ instead of $x \in Lim'_{[a,b]}(t), t \in (a,b]$ etc. The same conventions are true for the sets of continuous functions and these agree with the last paragraph from 2.10.

6. A Comparison Between Some Theorems Relative to Real and Pseudoboolean Differentiable Functions

6.1 Our purpose is to reproduce theorems relative to real functions, the way they are stated in the monography "Analiza matematica" by Miron Nicolescu, ed. tehnica, Bucuresti, 1958, volume II and then to give the same theorem in the pseudoboolean variant. The comparison will show the analogies between the two theories and the table 6.1 will be understood in the following manner: any attribute from the left is analogue to any attribute from the right.

real functions	*pseudoboolean functions*
continuous	*constant, continuous*
derivable	*with (left, right) limit*
differentiable	*differentiable*

table 6.1

We shall avoid sometimes to write explicitly the dual results.

6.2 ***Theorem** (page 328) Any real, monotonous increasing function* $f:[a,b]\to \mathbf{R}$ *on* $[a,b]$ *is derivable almost everywhere on this compact.*

Theorem Let $x:[a,b]\to \mathbf{B}_2$. The next statements are equivalent:

i) x is increasing

ii) $x(a)\le x(b)$, $x\in Diff_{[a,b]}$ and

$$|\{t\,|\,t\in (a,b], D'x(t)=1\}|+|\{t\,|\,t\in [a,b), D^*x(t)=1\}|\in \{0,1\}$$

6.3 ***Theorem** (Fubini, page 334) If* $\Sigma u_n(x)$ *is a convergent series in* $[a,b]$ *of monotonous functions of the same sense and if* $s(x)$ *is the sum of the series, then we have almost everywhere on* $[a,b]$*:*

$$\Sigma u_n^{'}(x)=s'(x)$$

Theorem If the sum $\underset{n\in \mathbf{N}}{\Xi}\, x_n(t)$ of monotonous functions $x_n:[a,b]\to \mathbf{B}_2, n\in \mathbf{N}$ is convergent and equal to $s(t)$, then

i) $\{n\,|\,x_n\not\equiv 0\}\in G_f$

ii) $s\in Diff_{[a,b]}$

iii) $D's(t)=\underset{n\in \mathbf{N}}{\Xi}\, D'x_n(t), t\in (a,b]$

6.4 ***Theorem** (Rolle, page 308) If the function* $f:[a,b]\to \mathbf{R}$

i) is continuous on the compact interval $[a,b]$

ii) it has in any point $x\in (a,b)$ *a (finite or infinite) derivative*

iii) $f(a)=f(b)$

then there exists a point $c\in (a,b)$ *so that* $f'(c)=0$

Theorem 1 Let $x:[a,b]\to \mathbf{B}_2$. The following statements are equivalent:

i) x is continuous (i.e. $x\in Cont_{[a,b]}\wedge Cont^{*}_{[a,b]}(a)\wedge Cont^{'}_{[a,b]}(b)$)

ii) x is constant

iii) $x\in Diff_{[a,b]}, D'x(t)=0, t\in (a,b]$ and $D^*x(t)=0, t\in [a,b)$

Theorem 2 Let $x:[a,b]\to \mathbf{B}_2$.

i) If x is continuous, then $x\in Lim_{[a,b]}, x(a)=x(b)$ and

$$Dx(t)=0, t\in (a,b)$$

ii) If $x\in Lim_{[a,b]}, \exists c\in (a,b)$ so that $x(a)=x(b)=x(c)$ and

$$Dx(t)=0, t\in (a,b)$$

then x is continuous.

Theorem 3 $Diff_{[a,b]} = Lim'_{[a,b]} \wedge Lim^*_{[a,b]}$ (see also 5.2).

Remark The differentiability is the analogue of the continuity here (left) and the existence of the limits is the analogue of the derivability (right).

6.5 ***Theorem*** *(Lagrange, page 308) If* $f:[a,b]\to \mathbf{R}$

i) is continuous on the compact interval $[a,b]$

ii) has a derivative in each point $x\in (a,b)$

then there exists a point $c\in (a,b)$ *so that*

$$f(b)-f(a)=(b-a)\cdot f'(c)$$

Theorem If $x\in Diff_{[a,b]}$, then one of the following statements is true:

$$\exists t\in (a,b], x(a)\oplus x(b)=D'x(t)$$
$$\exists t\in [a,b), x(a)\oplus x(b)=D^*x(t)$$

See also our theorems at 6.4

6.6 ***Theorem*** *(Scheefer, page 320) If* $I\subset \mathbf{R}$ *is an interval,* $f:I\to \mathbf{R}, g:I\to \mathbf{R}$ *are continuous and they admit, with the possible exception of an at most countable set* H *of points, finite and equal right superior derivative numbers, then* $f-g$ *is constant on* I.

Theorem If $x,y\in Diff_{[a,b]}$ satisfy

$$D'x(t)=D'y(t), t\in (a,b] \text{ and } D^*x(t)=D^*y(t), t\in [a,b)$$

then $x(t)=y(t)\oplus c, t\in [a,b]$.

Remarks This result is proved supposing that x,y are the restrictions at $[a,b]$ of $\mathbf{R}\to \mathbf{B}_2$ functions of the form 5.4 b.3). Counterexamples show that the weakening of the hypothesis is not possible.

6.7 ***Theorem*** *(Zygmund, page 318) Let* $f:I\to \mathbf{R}$, *where* $I\subset \mathbf{R}$ *is a possibly non compact interval, so that*

$$\limsup_{z\to x^-} f(z)\le f(x)\le \limsup_{u\to x^+} f(u)$$

in each point $x\in Int\, I$. *If the set* $f(E)$, *where*

$$E=\{x\mid x\in I, D^+f(x)\le 0\}$$

contains no interval, then f *is monotonous increasing in* I.

Theorem Let $x:[a,b]\to \mathbf{B}_2$ and the conditions:

i) $x(a)\le \overline{x(a+0)}, \overline{x(t-0)}\le x(t)\le \overline{x(t+0)}, t\in (a,b), \overline{x(b-0)}\le x(b)$

i)* $x(a)\le \underline{x(a+0)}, \underline{x(t-0)}\le x(t)\le \underline{x(t+0)}, t\in (a,b), \underline{x(b-0)}\le x(b)$

Each of i), i)* is equivalent to the fact that x is increasing.

The $\Rightarrow$ part of the proof consists in showing that the supposition:

there exist a',b' with $a\le a'<b'\le b$ so that $x(a')=1$ and $x(b')=0$

is a contradiction with the hypothesis i), or with the hypothesis i)*.

Remarks a) There surely exist versions of this theorem for the case when the domain of x is not compact.

b) We have a corollary, resulted by combining the theorem with its dual where the inequalities are reversed $\geq$ and x is decreasing:

i)' $\overline{D}' x(t) = 0, t \in (a,b]$ and $\overline{D}* x(t) = 0, t \in [a,b) \Leftrightarrow x \in Cont_{[a,b]}$

ii)' $\underline{D}' x(t) = 0, t \in (a,b]$ and $\underline{D}* x(t) = 0, t \in [a,b) \Leftrightarrow x \in Cont_{[a,b]}$

i.e. $\overline{Cont}'_{[a,b]} \wedge \overline{Cont}^*_{[a,b]} = Cont_{[a,b]}, \underline{Cont}'_{[a,b]} \wedge \underline{Cont}^*_{[a,b]} = Cont_{[a,b]}$

6.8 ***Theorem*** *(Dini, page 321) If one of the four derivative numbers of a function $f : I \to \mathbf{R}$ is a continuous function in the point $x \in Int\, I$, then f is derivable in that point.*

Theorem Let $x : [a,b] \to \mathbf{B}_2$ and $t \in (a,b)$. We have for $t \in (a,b)$:

i) $\overline{D}' x, \overline{D}* x \in Diff_{[a,b]}(t) \Leftrightarrow x \in Cont_{[a,b]}(t)$

ii) $\underline{D}' x, \underline{D}* x \in Diff_{[a,b]}(t) \Leftrightarrow x \in Cont_{[a,b]}(t)$

In order to prove the $\Rightarrow$ part of the theorem, we suppose at i) the existence of $\varepsilon > 0$ and $c \in \mathbf{B}_2$ so that $(t - \varepsilon, t) \subset [a,b]$ and $\forall \xi \in (t - \varepsilon, t), \overline{D}' x(\xi) = c$. It is shown that $c = 0$ and we take in consideration 6.7 (resulting that it is impossible to weaken the hypothesis).

6.9 ***Theorem*** *(Lebesgue, page 326) A function $f : [a,b] \to \mathbf{R}$ with bounded variation is almost everywhere derivable on the compact $[a,b]$.*

Theorem Let $x : [a,b] \to \mathbf{B}_2$. It is true the equivalence (see also our theorem at 6.2):

$$|\{t \mid t \in (a,b], \overline{D}' x(t) = 1\}|, |\{t \mid t \in [a,b), \overline{D}* x(t) = 1\}| \in G_f \Leftrightarrow x \in Diff_{[a,b]}$$

The $\Rightarrow$ part of the theorem is proved by showing, as a consequence of 6.7, that x is piecewise constant (piecewise continuous).

6.10 ***Theorem*** *(page 337) For any function $f : [a,b] \to \mathbf{R}$ with a bounded variation, we have almost everywhere:*

$$D(V_a^x f) = |Df(x)|$$

Theorem If $x : [a,b] \to \mathbf{B}_2$ has a finite left superior variation on $(a,b]$, then $\overline{V}'_{(a,\cdot]} x \in Diff'_{[a,b]}$ and the formula

$$D'(\overline{V}'_{(a,t]} x) = \overline{D}' x(t), t \in (a,b]$$

is true.

6.11 Let us finally recall that, accordingly to our intentions that were stated at 6.1, we have avoided to mention some dual results. For example, the missing result at 6.9 refers to the inferior derivatives.