On the generation of the asynchronous systems

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Abstract

The asynchronous circuits from the digital electrical engineering are modeled by the so-called asynchronous systems. An asynchronous system f is a multi-valued function that associates to each admissible input $u : \mathbf{R} \to \{0,1\}^m$ a set $f(u)$ of possible states $x \in f(u), x : \mathbf{R} \to \{0,1\}^n$. A special case of asynchronous system consists in the existence of a Boolean function $\Upsilon : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$ such that $\forall u, \forall x \in f(u)$, a certain equation involving Υ is fulfilled. Then Υ is called generator function (Moisil used the terminology of network function) and we say that f is generated by Υ.

Our purpose is to continue in a more general context the study of the generation of the asynchronous systems that was started in [5] with the autonomous systems.

1 Preliminaries

Notation 1 *Let be the arbitrary set M. The following notation will be useful:*

$$P^*(M) = \{M' | M' \subset M, M' \neq \emptyset\}.$$

Definition 2 *The set $\mathbf{B} = \{0,1\}$, endowed with the order $0 \leq 1$ and with the usual laws $^-, \cdot, \cup, \oplus$, is called the **binary Boole algebra**.*

Definition 3 *The **limit** $\lim\limits_{k \to \infty} \alpha^k \in \mathbf{B}^n$ of the sequence $\alpha : \mathbf{N} \to \mathbf{B}^n$, $\alpha(k) \overset{not}{=} \alpha^k, k \in \mathbf{N}$ is defined by*

$$\exists k' \in \mathbf{N}, \forall k'' \geq k', \alpha^{k''} = \lim_{k \to \infty} \alpha^k.$$

Definition 4 *Let be the function* $x : \mathbf{R} \to \mathbf{B}^n$*. Its* ***initial value*** $\lim_{t \to -\infty} x(t) \in \mathbf{B}^n$ *(also denoted by* $x(-\infty+0)$*) and its* ***final value*** $\lim_{t \to \infty} x(t) \in \mathbf{B}^n$ *(also denoted by* $x(\infty - 0)$*) are defined by*

$$\exists t' \in \mathbf{R}, \forall t'' \leq t', x(t'') = \lim_{t \to -\infty} x(t),$$

$$\exists t' \in \mathbf{R}, \forall t'' \geq t', x(t'') = \lim_{t \to \infty} x(t).$$

Notation 5 *We denote by* $\tau^d : \mathbf{R} \to \mathbf{R}$*,* $d \in \mathbf{R}$ *the* ***translation***

$$\forall t \in \mathbf{R}, \tau^d(t) = t - d.$$

Thus for any $x : \mathbf{R} \to \mathbf{B}^n$*, we denote by* $x \circ \tau^d : \mathbf{R} \to \mathbf{B}^n$ *the function*

$$\forall t \in \mathbf{R}, (x \circ \tau^d)(t) = x(t - d).$$

Definition 6 *The* ***characteristic function*** $\chi_A : \mathbf{R} \to \mathbf{B}$ *of the set* $A \subset \mathbf{R}$ *is given by*

$$\forall t \in \mathbf{R}, \chi_A(t) = \begin{cases} 1, t \in A \\ 0, \ else \end{cases}.$$

Notation 7 *We use the following notation*

$$Seq = \{(t_k) | t_k \in \mathbf{R}, k \in \mathbf{N}, \ t_0 < ... < t_k < ... \text{ is unbounded from above}\}.$$

Definition 8 *The* ***cyclic values*** $\mu \in \mathbf{B}^n$ *and the* ***co-cyclic values*** $\mu' \in \mathbf{B}^n$ *of* $x : \mathbf{R} \to \mathbf{B}^n$ *are defined by*

$$\exists (t_k) \in Seq, \forall k \in \mathbf{N}, x(t_k) = \mu,$$

$$\exists (t_k) \in Seq, \forall k \in \mathbf{N}, x(-t_k) = \mu'.$$

Remark 9 *The initial value and the final value of* $x : \mathbf{R} \to \mathbf{B}^n$ *may not exist and if they exist, then they are unique. The cyclic values and the co-cyclic values of* x *always exist and they are not unique in general. The existence of the final value (of the initial value) of* x *is equivalent with the existence of a unique cyclic value (of a unique co-cyclic value) of* x *and in this case the final value (the initial value) coincides with the unique cyclic value (with the unique co-cyclic value).*

Definition 10 *The set of the cyclic values of* x *is called the* ***limit cycle*** *of* x *and its notation is* $LC(x)$*. The set of the co-cyclic values of* x *is called the* ***limit co-cycle*** *of* x *and is denoted by* $LC^*(x)$*.*

Definition 11 *A function* $x : \mathbf{R} \to \mathbf{B}^n$ *is called* $n-$***signal****, shortly* ***signal*** *if* $\mu \in B^n$ *and* $(t_k) \in Seq$ *exist so that*

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... \quad (1)$$

where in (1) we have abusively used the same symbols $\cdot, \oplus$ *for the laws that are induced by those of* $\mathbf{B}$*. The set of the* $n-$*signals is denoted by* $S^{(n)}$ *and instead of* $S^{(1)}$ *we usually write* S*.*

Remark 12 *Normally we have* $S^{(n)} \times S^{(m)} = \{(x,u)|(x,u) : \mathbf{R} \times \mathbf{R} \to \mathbf{B}^n \times \mathbf{B}^m, x \in S^{(n)}, u \in S^{(m)}\}$*. In this paper we need to work with another type of Cartesian product of spaces of functions, denoted by '*$\times$*' also, such that*

$$S^{(n)} \times S^{(m)} = \{(x,u)|(x,u) : \mathbf{R} \to \mathbf{B}^n \times \mathbf{B}^m, x \in S^{(n)}, u \in S^{(m)}\}.$$

We keep in mind this fact when writing $S^{(n)}$ *instead of* S^n*. This remark on the way that the Cartesian product of spaces of functions is made is obviously justified by the existence of a unique time variable and between the consequences derived from here we have the identifications* $S^{(n)} \times S^{(m)} = S^{(n+m)}$ *and* $P^*(S^{(n)}) \times P^*(S^{(m)}) = P^*(S^{(n+m)})$*.*

Definition 13 *Let* $x \in S^{(n)}$ *be given by (1). Its* ***left limit*** $x(t-0)$ *is the* $\mathbf{R} \to \mathbf{B}^n$ *function defined as*

$$x(t-0) = \mu \cdot \chi_{(-\infty,t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0,t_1]}(t) \oplus ... \oplus x(t_k) \cdot \chi_{(t_k,t_{k+1}]}(t) \oplus ... \quad (2)$$

Definition 14 *Let be* $U \in P^*(S^{(m)})$*. A multi-valued function* $f : U \to P^*(S^{(n)})$ *is called* ***asynchronous system****, shortly* ***system****. Any* $u \in U$ *is called (****admissible****)* ***input*** *and the functions* $x \in f(u)$ *are called (****possible****)* ***states****.*

Definition 15 *The system* f *is called a* ***subsystem*** *of* $g : V \to P^*(S^{(n)})$*,* $V \in P^*(S^{(m)})$ *and we write* $f \subset g$ *if*

$$U \subset V \text{ and } \forall u \in U, f(u) \subset g(u).$$

Remark 16 *The concept of system originates in the modeling of the asynchronous circuits. The multi-valued character of the cause-effect association is due to the statistical fluctuations in the fabrication process, the variations in the ambiental temperature, the power supply etc.*

Sometimes the systems are given by equations and/or inequalities.

We interpret $f \subset g$ *in the following way: the systems* f *and* g *model the same circuit, but the model represented by* f *is more precise than the model represented by* g*.*

Definition 17 *If* $\forall u \in U$*, the set* $f(u)$ *has exactly one element, then the system* f *is called* ***deterministic*** *and we use the notation* $f : U \to S^{(n)}$ *of the usual functions.*

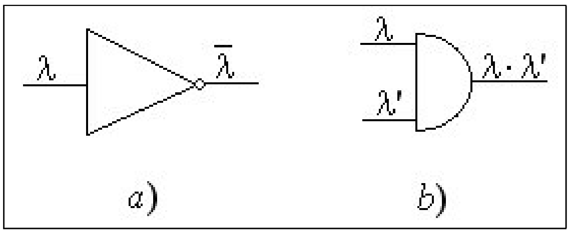

Figure 1: The logical gate NOT and the logical gate AND

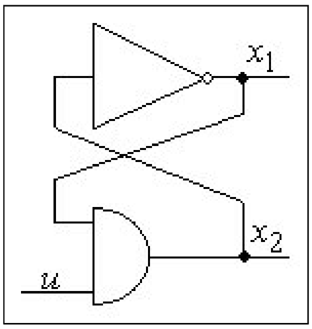

Figure 2: Example of circuit with feedback

2 An example

Example 18 *The logical gates from Figure 1 compute for any* $\lambda, \lambda' \in \mathbf{B}$ *the complement* $\mathbf{B} \ni \lambda \to \overline{\lambda} \in \mathbf{B}$ *and the product* $\mathbf{B} \times \mathbf{B} \ni (\lambda, \lambda') \to \lambda \cdot \lambda' \in \mathbf{B}$. *This computation is made with a nonconstant and generally unknown delay. We are interested in modeling the circuit from Figure 2 where the way that the function*

$$\Upsilon : \mathbf{B}^2 \times \mathbf{B} \to \mathbf{B}^2, \mathbf{B}^2 \times \mathbf{B} \ni (\mu, \nu) \to \Upsilon(\mu, \nu) = (\overline{\mu}_2, \nu \cdot \mu_1) \in \mathbf{B}^2$$

is computed is indicated by $\rho : \mathbf{R} \to \mathbf{B}^2$,

$$\rho(t) = (1,0) \cdot \chi_{\{2\}}(t) \oplus (1,1) \cdot \chi_{\{4\}}(t) \oplus (0,1) \cdot \chi_{\{5\}}(t) \oplus$$

$$\oplus (0,1) \cdot \chi_{\{7\}}(t) \oplus (1,1) \cdot \chi_{\{8\}}(t) \oplus (1,1) \cdot \chi_{\{9\}}(t) \oplus (1,1) \cdot \chi_{\{10\}}(t) \oplus ...$$

The function ρ *shows the following behavior of the circuit: at* $t = 2$ *the coordinate* Υ_1 *is computed, at* $t = 4$ *both coordinates* Υ_1 *and* Υ_2 *are computed, at* $t = 5$ *only* Υ_2 *is computed, at* $t = 7$ *the coordinate* Υ_2 *is computed again and at* $t \in \{8, 9, 10, ...\}$ *both* Υ_1 *and* Υ_2 *are computed. At* $t \in \mathbf{R} \setminus \{2, 4, 5, 7, 8, 9, 10, ...\}$ *the circuit does not make any computation and keeps the previous values of the state* $x \in S^{(2)}$.

The initial state $\mu \in \mathbf{B}^2$ *and the input* $u \in S$ *are*

$$\mu = (0,0),$$

$$u(t) = \chi_{[2,4)}(t) \oplus \chi_{[6,\infty)}(t).$$

We have

$t < 2$:

$$(x_1(t), x_2(t)) = (0,0),$$

$t \in [2,4)$:

$$(x_1(t), x_2(t)) =$$

$$= (\Upsilon_1(x_1(2-0), x_2(2-0), u(2-0)), x_2(2-0))$$
$$= (\Upsilon_1(0,0,0), 0) = (\overline{0}, 0) = (1,0),$$

$t \in [4,5)$:

$$(x_1(t), x_2(t)) =$$

$$= (\Upsilon_1(x_1(4-0), x_2(4-0), u(4-0)), \Upsilon_2(x_1(4-0), x_2(4-0), u(4-0)))$$
$$= (\Upsilon_1(1,0,1), \Upsilon_2(1,0,1)) = (\overline{0}, 1 \cdot 1) = (1,1),$$

$t \in [5,7)$:

$$(x_1(t), x_2(t)) =$$

$$= (x_1(5-0), \Upsilon_2(x_1(5-0), x_2(5-0), u(5-0)))$$
$$= (1, \Upsilon_2(1,1,0)) = (1, 0 \cdot 1) = (1,0),$$

$t \in [7,8)$:

$$(x_1(t), x_2(t)) =$$

$$= (x_1(7-0), \Upsilon_2(x_1(7-0), x_2(7-0), u(7-0)))$$
$$= (1, \Upsilon_2(1,0,1)) = (1, 1 \cdot 1) = (1,1),$$

$$...$$

After a few more similar computations we get that the state $x(t)$ *is given by*

$$x(t) = (1,0) \cdot \chi_{[2,4)}(t) \oplus (1,1) \cdot \chi_{[4,5)}(t) \oplus (1,0) \cdot \chi_{[5,7)}(t) \oplus$$

$$\oplus (1,1) \cdot \chi_{[7,8) \cup [11,12) \cup ...}(t) \oplus (0,1) \cdot \chi_{[8,9) \cup [12,13) \cup ...}(t) \oplus (1,0) \cdot \chi_{[10,11) \cup [14,15) \cup ...}(t)$$

Of course that at this level of generality of the exposure, in order to describe all the possibilities of work of the circuit from Figure 2, infinitely many functions ρ *exist giving the way that* Υ *is computed.*

3 Progressive sequences and progressive functions

Definition 19 *The sequence* $\alpha : \mathbf{N} \to \mathbf{B}^n, \alpha(k) \overset{not}{=} \alpha^k, k \in \mathbf{N}$ *is **progressive** if* $\forall i \in \{1, ..., n\}$, *the set*

$$\{k | k \in \mathbf{N}, \alpha_i^k = 1\}$$

is infinite. The set of the $\mathbf{N} \to \mathbf{B}^n$ *progressive sequences is denoted by* Π_n.

Definition 20 *The function* $\rho : \mathbf{R} \to \mathbf{B}^n$ *is called **progressive**, if* $\alpha \in \Pi_n$ *and* $(t_k) \in Seq$ *exist such that*

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{3}$$

The set of the progressive functions is denoted by P_n.

Remark 21 *In the previous two definitions, the so-called progress condition that all the sets* $\{k | k \in \mathbf{N}, \alpha_i^k = 1\}$ *are infinite,* $i = \overline{1, n}$ *expresses the idea that the coordinates of some Boolean function are computed countably many times, thus their computation time is arbitrary, finite, possibly variable (depending on manufacturing fluctuations in delay related parameters, on the temperature, on the tension of the mains etc.) This was anticipated at Example 18.*

Some properties of invariance are interesting in this context, of the form $..., \forall \rho \in P_n, ...$ *meaning that for each manufactured instance of a design, for each admissible temperature and for each admissible tension of the mains, we restrict our attention to the information that is common to all of them. Such properties occur in [5].*

4 Basic definitions

Definition 22 *Let be the function* $\Upsilon : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n, \Upsilon = (\Upsilon_1, ..., \Upsilon_n)$. *For* $\lambda \in \mathbf{B}^n, \lambda = (\lambda_1, ..., \lambda_n)$ *we define the function* $\Upsilon^\lambda : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$, *called* Υ ***at the power*** λ, *or the* $\lambda-$***iterate*** *of* Υ, *by* $\forall \mu \in \mathbf{B}^n, \forall \nu \in \mathbf{B}^m$,

$$\Upsilon^\lambda(\mu, \nu) = (\overline{\lambda_1} \cdot \mu_1 \oplus \lambda_1 \cdot \Upsilon_1(\mu, \nu), ..., \overline{\lambda_n} \cdot \mu_n \oplus \lambda_n \cdot \Upsilon_n(\mu, \nu)).$$

Remark 23 *In the previous definition the role of* λ *is that of showing, in a computation that starts in* μ *and heads towards* $\Upsilon(\mu, \nu)$, *which coordinate* Υ_i *of* Υ *is computed: if* $\lambda_i = 0$, *then* $\Upsilon_i^\lambda(\mu, \nu) = \mu_i$ *and* Υ_i *is not computed, while if* $\lambda_i = 1$, *then* $\Upsilon_i^\lambda(\mu, \nu) = \Upsilon_i(\mu, \nu)$ *and* Υ_i *is computed,* $i = \overline{1, n}$.

Definition 24 *Let be the progressive sequence* $\alpha \in \Pi_n$*. We define the functions* $\Upsilon^{\alpha^0 ... \alpha^k} : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n, k \in \mathbf{N}$ *by* $\forall k \in \mathbf{N}, \forall \mu \in \mathbf{B}^n, \forall \nu \in \mathbf{B}^m$,

$$\Upsilon^{\alpha^0 ... \alpha^k \alpha^{k+1}}(\mu, \nu) = \Upsilon^{\alpha^{k+1}}(\Upsilon^{\alpha^0 ... \alpha^k}(\mu, \nu), \nu). \tag{4}$$

Definition 25 *Let be* $\rho \in P_n$,

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{5}$$

where $\alpha \in \Pi_n$ *and* $(t_k) \in Seq$*. For any* $u \in S^{(m)}$,

$$u(t) = u^{-1} \cdot \chi_{(-\infty, \psi_0)}(t) \oplus u^0 \cdot \chi_{[\psi_0, \psi_1)}(t) \oplus ... \oplus u^k \cdot \chi_{[\psi_k, \psi_{k+1})}(t) \oplus ...$$

with $u^{-1}, u^0, ..., u^k, ... \in \mathbf{B}^m, (\psi_k) \in Seq$ *we suppose without loss of generality the existence of the indexes* $p_0, p_1, p_2, ... \in \mathbf{N}$ *such that*

$$t_0 < t_1 < ... < t_{p_0} \leq \psi_0 < t_{p_0+1} < t_{p_0+2} < ... \tag{6}$$

$$... < t_{p_1} \leq \psi_1 < t_{p_1+1} < t_{p_1+2} < ... < t_{p_2} \leq \psi_2 < ...$$

For this, we can choose in (5) conveniently some sequence $(t'_k) \in Seq, (t'_k) \supset (t_k)$ *and we can supplement* α *with null terms to some sequence* $\alpha' \in \Pi_n$*. We define* Υ ***at the power*** $-\rho$, $\Upsilon^{-\rho} : \mathbf{R} \times \mathbf{B}^n \times S^{(m)} \to \mathbf{B}^n$ *by*

$$\begin{aligned} \mu^0 &= \Upsilon^{\alpha^0 ... \alpha^{p_0}}(\mu, u^{-1}), \\ \mu^1 &= \Upsilon^{\alpha^{p_0+1} ... \alpha^{p_1}}(\mu^0, u^0), \\ \mu^2 &= \Upsilon^{\alpha^{p_1+1} ... \alpha^{p_2}}(\mu^1, u^1), \\ &... \end{aligned} \tag{7}$$

and $\forall t \in \mathbf{R}, \forall \mu \in \mathbf{B}^n$,

$$\Upsilon^{-\rho}(t, \mu, u) = \tag{8}$$

$$= \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \Upsilon^{\alpha^0}(\mu, u^{-1}) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus \Upsilon^{\alpha^0 ... \alpha^{p_0}}(\mu, u^{-1}) \cdot \chi_{[t_{p_0}, t_{p_0+1})}(t) \oplus$$

$$\oplus \Upsilon^{\alpha^{p_0+1}}(\mu^0, u^0) \cdot \chi_{[t_{p_0+1}, t_{p_0+2})}(t) \oplus ... \oplus \Upsilon^{\alpha^{p_0+1} ... \alpha^{p_1}}(\mu^0, u^0) \cdot \chi_{[t_{p_1}, t_{p_1+1})}(t) \oplus$$

$$\oplus \Upsilon^{\alpha^{p_1+1}}(\mu^1, u^1) \cdot \chi_{[t_{p_1+1}, t_{p_1+2})}(t) \oplus ... \oplus \Upsilon^{\alpha^{p_1+1} ... \alpha^{p_2}}(\mu^1, u^1) \cdot \chi_{[t_{p_2}, t_{p_2+1})}(t) \oplus ...$$

Similarly, we replace (6) by

$$t_0 < t_1 < ... < t_{p_0} < \psi_0 \leq t_{p_0+1} < t_{p_0+2} < ... \tag{9}$$

$$... < t_{p_1} < \psi_1 \leq t_{p_1+1} < t_{p_1+2} < ... < t_{p_2} < \psi_2 \leq ...$$

and we define Υ ***at the power*** ρ, $\Upsilon^{\rho} : \mathbf{R} \times \mathbf{B}^n \times S^{(m)} \to \mathbf{B}^n$ *by (7) and* $\forall t \in \mathbf{R}, \forall \mu \in \mathbf{B}^n$,

$$\Upsilon^{\rho}(t, \mu, u) = \tag{10}$$

$$= \mu\cdot\chi_{(-\infty,t_0)}(t)\oplus\Upsilon^{\alpha^0}(\mu,u^{-1})\cdot\chi_{[t_0,t_1)}(t)\oplus...\oplus\Upsilon^{\alpha^0...\alpha^{p_0}}(\mu,u^{-1})\cdot\chi_{[t_{p_0},t_{p_0+1})}(t)\oplus$$

$$\oplus\Upsilon^{\alpha^{p_0+1}}(\mu^0,u^0)\cdot\chi_{[t_{p_0+1},t_{p_0+2})}(t)\oplus...\oplus\Upsilon^{\alpha^{p_0+1}...\alpha^{p_1}}(\mu^0,u^0)\cdot\chi_{[t_{p_1},t_{p_1+1})}(t)\oplus$$

$$\oplus\Upsilon^{\alpha^{p_1+1}}(\mu^1,u^1)\cdot\chi_{[t_{p_1+1},t_{p_1+2})}(t)\oplus...\oplus\Upsilon^{\alpha^{p_1+1}...\alpha^{p_2}}(\mu^1,u^1)\cdot\chi_{[t_{p_2},t_{p_2+1})}(t)\oplus...$$

Example 26 $\Upsilon:\mathbf{B}\times\mathbf{B}\to\mathbf{B},\forall\mu\in\mathbf{B},\forall\nu\in\mathbf{B},\Upsilon(\mu,\nu)=\mu\cdot\overline{\nu}$ *and we take*

$$\rho(t)=\chi_{\{0,1,2,...\}}(t),$$
$$\rho'(t)=\chi_{\{1,2,3,...\}}(t).$$

We have:

$$\Upsilon^{-\rho}(t,\mu,\chi_{[0,\infty)})=\mu\cdot\chi_{(-\infty,1)}(t),$$
$$\Upsilon^{\rho}(t,\mu,\chi_{[0,\infty)})=\mu\cdot\chi_{(-\infty,0)}(t),$$
$$\Upsilon^{-\rho'}(t,\mu,\chi_{[0,\infty)})=\mu\cdot\chi_{(-\infty,1)}(t)=\Upsilon^{\rho'}(t,\mu,\chi_{[0,\infty)}).$$

Remark 27 $\Upsilon^{-\rho}$ *makes use of the values* $u(t_k-0),k\in\mathbf{N}$ *and* Υ^{ρ} *makes use of the values* $u(t_k),k\in\mathbf{N}$. *In the previous example,* $\Upsilon^{-\rho}(t,\mu,\chi_{[0,\infty)})\neq\Upsilon^{\rho}(t,\mu,\chi_{[0,\infty)})$ *because* $\rho(0)\neq 0$, *thus* Υ *is computed in* $t=0$ *and* $t=0$ *is also a discontinuity point of* $u=\chi_{[0,\infty)}:u(0-0)=0,u(0)=1$. *On the other hand* $\Upsilon^{-\rho'}(t,\mu,\chi_{[0,\infty)})=\Upsilon^{\rho'}(t,\mu,\chi_{[0,\infty)})$ *because in the only discontinuity point* $t=0$ *of* u *we have* $\rho'(0)=0$, *thus* Υ *is not computed in* $t=0$ *and the first time instant where the complement is computed is* $t=1$.

The occurrence of $\Upsilon^{-\rho}(t,\mu,u)$ *and of* $\Upsilon^{\rho}(t,\mu,u)$ *corresponds to two different concepts of causality, namely the one when only the previous values of the cause influence the present value of the effect, respectively the one when the present value of the cause influences also the present value of the effect.*

Definition 28 *The functions* $\Upsilon^{-\rho}(\cdot,\mu,u),\Upsilon^{\rho}(\cdot,\mu,u):\mathbf{R}\to\mathbf{B}^n$, *where* $\mu\in\mathbf{B}^n,u\in S^{(m)}$ *and* $\rho\in P_n$, *are called the* $-\rho-$***motion*** *and the* $\rho-$***motion*** *of the point* μ *under the input* u.

Definition 29 *The quadruples* $\upsilon^-=(\mathbf{R},\mathbf{B}^n,\mathbf{B}^m,(\Upsilon^{-\rho})_{\rho\in P_n})$, $\upsilon=(\mathbf{R},\mathbf{B}^n,\mathbf{B}^m,(\Upsilon^{\rho})_{\rho\in P_n})$ *are called* ***Boolean dynamical systems****.* $\mathbf{R}$ *is the* ***time set*** *and* $t\in\mathbf{R}$ *is the* ***time variable****.* $\mathbf{B}^n$ *is called the* ***state space*** *and its points* $\mu\in\mathbf{B}^n$ *are called* ***states***[1]*.* $\mathbf{B}^m$ *has the name of* ***input space****. The function* Υ *is called the* ***generator function*** *of* υ^-,υ *and* $\Upsilon^{-\rho},\Upsilon^{\rho},\rho\in P_n$ *are called the* ***computations*** *of* Υ.

[1]We abusively identify a function $x\in S^{(n)}$ normally called state with its values $\mu=x(t)$.

5 The generation of the asynchronous systems

Definition 30 *Let be* $\Upsilon : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$*. The systems* $\Sigma_\Upsilon^-, \Sigma_\Upsilon : S^{(m)} \to P^*(S^{(n)})$,

$$\forall u \in S^{(m)}, \Sigma_\Upsilon^-(u) = \{\Upsilon^{-\rho}(\cdot, \mu, u) | \mu \in \mathbf{B}^n, \rho \in P_n\},$$

$$\forall u \in S^{(m)}, \Sigma_\Upsilon(u) = \{\Upsilon^{\rho}(\cdot, \mu, u) | \mu \in \mathbf{B}^n, \rho \in P_n\}$$

are called the ***universal regular asynchronous systems*** *that are* ***generated*** *by* Υ*. The function* Υ *is called the* ***generator function*** *of* $\Sigma_\Upsilon^-, \Sigma_\Upsilon$.

Definition 31 *A system* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *is called* ***regular*** *if* $\Upsilon : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$ *exists such that one of* $f \subset \Sigma_\Upsilon^-, f \subset \Sigma_\Upsilon$ *is true. In this situation the function* Υ *is called the* ***generator function*** *of* f.

Remark 32 *In the previous definitions, the attribute 'universal' means maximal relative to the inclusion and the attribute 'regular' means the existence of a generator function* Υ.

While the system Σ_Υ^- *is associated to the Boolean dynamical system* $\upsilon^- = (\mathbf{R}, \mathbf{B}^n, \mathbf{B}^m, (\Upsilon^{-\rho})_{\rho \in P_n})$*, its subsystems* $f \subset \Sigma_\Upsilon^-$ *may result by requesting that the initial states* μ *run over a subset of* $\mathbf{B}^n$ *(initial conditions), the inputs* u *run over a subset* U *of* $S^{(m)}$ *(conditions of admissibility of the inputs), or perhaps* $\Upsilon^{-\rho}$ *run over a subset of all the computations of* Υ *(restrictions imposed on the computation time of the coordinate functions* $\Upsilon_i, i = \overline{1,n}$*). Similarly for* Σ_Υ.

In general the regular asynchronous systems do not have a unique generator function, for example

$$\exists \Upsilon, \Upsilon' : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n, \Upsilon \neq \Upsilon' \text{ and } \exists u \in S^{(m)}, \Sigma_\Upsilon^-(u) \cap \Sigma_{\Upsilon'}^-(u) \neq \emptyset.$$

6 The connection between dynamical systems, asynchronous systems and equations

Theorem 33 *We consider the point* $\mu \in \mathbf{B}^n$ *and the functions* $u \in S^{(m)}, \rho \in P_n$*. The equation*

$$\begin{cases} x(-\infty + 0) = \mu \\ x(t) = \Upsilon^{\rho(t)}(x(t-0), u(t-0)) \end{cases}$$

has the unique solution

$$x(t) = \Upsilon^{-\rho}(t, \mu, u)$$

and the equation

$$\begin{cases} x(-\infty+0)=\mu \\ x(t)=\Upsilon^{\rho(t)}(x(t-0),u(t)) \end{cases}$$

has the unique solution

$$x(t)=\Upsilon^{\rho}(t,\mu,u).$$

Proof. We prove the first statement of the theorem and let be

$$\rho(t)=\alpha^0\cdot\chi_{\{t_0\}}(t)\oplus...\oplus\alpha^k\cdot\chi_{\{t_k\}}(t)\oplus... \tag{11}$$

where $\alpha\in\Pi_n$ and $(t_k)\in Seq$. We can write

$$u(t)=u^{-1}\cdot\chi_{(-\infty,\psi_0)}(t)\oplus u^0\cdot\chi_{[\psi_0,\psi_1)}(t)\oplus...\oplus u^k\cdot\chi_{[\psi_k,\psi_{k+1})}(t)\oplus...$$

for some $u^{-1},u^0,...,u^k,...\in\mathbf{B}^m$ and $(\psi_k)\in Seq$. We suppose that the sequence $p_0,p_1,p_2,...\in\mathbf{N}$ exists such that

$$t_0<t_1<...<t_{p_0}\le\psi_0<t_{p_0+1}<t_{p_0+2}<... \tag{12}$$

$$...<t_{p_1}\le\psi_1<t_{p_1+1}<t_{p_1+2}<...<t_{p_2}\le\psi_2<...$$

is true. We have from the first equation:

$t<t_0:\quad x(t)=x(t-0)=x(-\infty+0)=\mu,$

$t=t_0:\quad x(t_0)=\Upsilon^{\alpha^0}(\mu,u^{-1}),$

$t\in(t_0,t_1):\quad x(t)=x(t-0)=\Upsilon^{\alpha^0}(\mu,u^{-1}),$

$t=t_1:\quad x(t_1)=\Upsilon^{\alpha^1}(\Upsilon^{\alpha^0}(\mu,u^{-1}),u^{-1})=\Upsilon^{\alpha^0\alpha^1}(\mu,u^{-1}),$

...

$t=t_{p_0}:\quad x(t_{p_0})=\Upsilon^{\alpha^0...\alpha^{p_0}}(\mu,u^{-1})\overset{not}{=}\mu^0,$

$t\in(t_{p_0},t_{p_0+1}):\quad x(t)=x(t-0)=\mu^0,$

$t=t_{p_0+1}:\quad x(t_{p_0+1})=\Upsilon^{\alpha^{p_0+1}}(\mu^0,u^0),$

$t\in(t_{p_0+1},t_{p_0+2}):\quad x(t)=x(t-0)=\Upsilon^{\alpha^{p_0+1}}(\mu^0,u^0),$

...

$t=t_{p_1}:\quad x(t_{p_1})=\Upsilon^{\alpha^{p_0+1}...\alpha^{p_1}}(\mu^0,u^0)\overset{not}{=}\mu^1,$

$t\in(t_{p_1},t_{p_1+1}):\quad x(t)=x(t-0)=\mu^1,$

$t=t_{p_1+1}:\quad x(t_{p_1+1})=\Upsilon^{\alpha^{p_1+1}}(\mu^1,u^1),$

$t\in(t_{p_1+1},t_{p_1+2}):\quad x(t)=x(t-0)=\Upsilon^{\alpha^{p_1+1}}(\mu^1,u^1),$

...

This completes the proof of the first part of the theorem.

We replace now (12) with

$$t_0 < t_1 < ... < t_{p_0} < \psi_0 \leq t_{p_0+1} < t_{p_0+2} < ... \tag{13}$$

$$... < t_{p_1} < \psi_1 \leq t_{p_1+1} < t_{p_1+2} < ... < t_{p_2} < \psi_2 \leq ...$$

We prove by following the same steps like previously that $\Upsilon^{\rho}(t, \mu, u)$ is the unique solution of the second equation. ■

Remark 34 *We can associate to the equations from Theorem 33, when μ runs in $\mathbf{B}^n$, u runs in $S^{(m)}$ and ρ runs in P_n, the dynamical systems $\upsilon^- = (\mathbf{R}, \mathbf{B}^n, \mathbf{B}^m, (\Upsilon^{-\rho})_{\rho \in P_n})$, $\upsilon = (\mathbf{R}, \mathbf{B}^n, \mathbf{B}^m, (\Upsilon^{\rho})_{\rho \in P_n})$ and the asynchronous systems $\Sigma_{\Upsilon}^-, \Sigma_{\Upsilon}$.*

7 Equivalencies

Definition 35 *Let be the functions $\Upsilon, \Upsilon' : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$. We say that the Boolean dynamical systems $\upsilon^- = (\mathbf{R}, \mathbf{B}^n, \mathbf{B}^m, (\Upsilon^{-\rho})_{\rho \in P_n})$, $\upsilon'^- = (\mathbf{R}, \mathbf{B}^n, \mathbf{B}^m, (\Upsilon'^{-\rho})_{\rho \in P_n})$, the Boolean dynamical systems $\upsilon = (\mathbf{R}, \mathbf{B}^n, \mathbf{B}^m, (\Upsilon^{\rho})_{\rho \in P_n})$, $\upsilon' = (\mathbf{R}, \mathbf{B}^n, \mathbf{B}^m, (\Upsilon'^{\rho})_{\rho \in P_n})$, the asynchronous systems $\Sigma_{\Upsilon}^-, \Sigma_{\Upsilon'}^-$ and the asynchronous systems $\Sigma_{\Gamma}, \Sigma_{\Gamma'}$ are* ***equivalent****, if the bijections $G : \mathbf{B}^m \to \mathbf{B}^m$, $H : \mathbf{B}^n \to \mathbf{B}^n$ exist such that the following diagram is commutative*

$$\begin{array}{ccc} \mathbf{B}^n \times \mathbf{B}^m & \overset{\Upsilon}{\to} & \mathbf{B}^n \\ H \times G \downarrow & & \downarrow H \\ \mathbf{B}^n \times \mathbf{B}^m & \overset{\Upsilon'}{\to} & \mathbf{B}^n \end{array}$$

If this is true, we say that G and H ***transform the generator function*** *Υ* ***in the generator function*** *Υ'.*

Remark 36 *This definition concerning the equivalence refers to two changes of the systems of coordinates, made on $\mathbf{B}^m$ by G and on $\mathbf{B}^n$ by H.*

Definition 37 *Let be $\mu \in \mathbf{B}^n, u \in S^{(m)}$ and the functions $\rho, \rho' \in P_n$. The equations*

$$\begin{cases} x(-\infty + 0) = \mu \\ x(t) = \Upsilon^{\rho(t)}(x(t-0), u(t-0)) \end{cases}, \tag{14}$$

$$\begin{cases} y(-\infty + 0) = \mu \\ y(t) = \Upsilon^{\rho'(t)}(y(t-0), u(t-0)) \end{cases} \tag{15}$$

and the motions $\Upsilon^{-\rho}(\cdot, \mu, u)$ and $\Upsilon^{-\rho'}(\cdot, \mu, u)$ are ***equivalent*** *if the bijective, continuous strictly increasing function $h : \mathbf{R} \to \mathbf{R}$ exists such that*

$$\Upsilon^{-\rho'}(t, \mu, u) = \Upsilon^{-\rho}(h(t), \mu, u). \tag{16}$$

Similarly, the equations

$$\begin{cases} x(-\infty + 0) = \mu \\ x(t) = \Upsilon^{\rho(t)}(x(t-0), u(t)) \end{cases}, \tag{17}$$

$$\begin{cases} y(-\infty + 0) = \mu \\ y(t) = \Upsilon^{\rho'(t)}(y(t-0), u(t)) \end{cases} \tag{18}$$

and the motions $\Upsilon^{\rho}(\cdot, \mu, u)$ *and* $\Upsilon^{\rho'}(\cdot, \mu, u)$ *are* ***equivalent*** *if the bijective, continuous strictly increasing function* $h : \mathbf{R} \to \mathbf{R}$ *exists such that*

$$\Upsilon^{\rho'}(t, \mu, u) = \Upsilon^{\rho}(h(t), \mu, u). \tag{19}$$

Remark 38 *The previous definition states that the solutions* $\Upsilon^{-\rho}(\cdot, \mu, u)$*,* $\Upsilon^{-\rho'}(\cdot, \mu, u)$ *of (14),(15) are equivalent if they are equal functions regardless the time flow, which is given by* t *for* $\Upsilon^{-\rho'}(\cdot, \mu, u)$ *and by* $h(t)$ *for* $\Upsilon^{-\rho}(\cdot, \mu, u)$*. If the function* $h : \mathbf{R} \to \mathbf{R}$ *is bijective, continuous and strictly increasing, then* h^{-1} *has the same properties [5], thus Definition 37 is that of an equivalence relation indeed. We have the sufficient condition* $\rho' = \rho \circ h$ *in order that (16) is true [5]. The same interpretation of the equivalence follows for the motions* $\Upsilon^{\rho}(\cdot, \mu, u)$*,* $\Upsilon^{\rho'}(\cdot, \mu, u)$ *and the equations (17), (18).*

Definition 39 *Let be* $\mu \in \mathbf{B}^n, u \in S^{(m)}$ *and* $\rho, \rho' \in P_n$*. The equations (14), (15) and the motions* $\Upsilon^{-\rho}(\cdot, \mu, u)$*,* $\Upsilon^{-\rho'}(\cdot, \mu, u)$ *are* ***equivalent*** *if*

$$LC(\Upsilon^{-\rho}(\cdot, \mu, u)) = LC(\Upsilon^{-\rho'}(\cdot, \mu, u)).$$

Similarly, the equations (17), (18) and the motions $\Upsilon^{\rho}(\cdot, \mu, u)$*,* $\Upsilon^{\rho'}(\cdot, \mu, u)$ *are* ***equivalent*** *if*

$$LC(\Upsilon^{\rho}(\cdot, \mu, u)) = LC(\Upsilon^{\rho'}(\cdot, \mu, u)).$$

Remark 40 *In Definition 39, the motions* $\Upsilon^{-\rho}(\cdot, \mu, u)$*,* $\Upsilon^{-\rho'}(\cdot, \mu, u)$ *and the motions* $\Upsilon^{\rho}(\cdot, \mu, u)$*,* $\Upsilon^{\rho'}(\cdot, \mu, u)$ *are equivalent if they start from the same initial value* μ *and, running under the same input* $u \in S^{(m)}$*, they reach the same limit cycle. The equivalence classes of* $\Upsilon^{-\rho}(\cdot, \mu, u)$ *and* $\Upsilon^{\rho}(\cdot, \mu, u)$ *have the property that the unique limit cycle reached by all their elements depends on* μ *and* u *only and it does not depend on* ρ*.*

8 Huffman systems

Definition 41 *The system* h *is called* ***Huffman*** *if it fulfills one of the next two conditions a), b):*

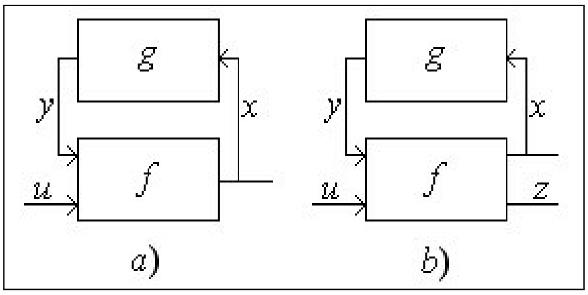

Figure 3: Huffman systems

a) $h : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$; *the function* $\Upsilon : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$ *and the systems* $f : S^{(n)} \times U \to P^*(S^{(n)}), g : S^{(n)} \to P^*(S^{(n)})$ *exist such that*

$$\forall (y,u) \in S^{(n)} \times U, \exists \lim_{t\to\infty} \Upsilon(y(t),u(t)) \Longrightarrow \Longrightarrow \forall x \in f(y,u), \lim_{t\to\infty} x(t) = \lim_{t\to\infty} \Upsilon(y(t),u(t)), \tag{20}$$

$$\forall x \in S^{(n)}, \exists \lim_{t\to\infty} x(t) \Longrightarrow \forall y \in g(x), \lim_{t\to\infty} y(t) = \lim_{t\to\infty} x(t), \tag{21}$$

$$\forall u \in U, h(u) = \{x | \exists y \in S^{(n)}, x \in f(y,u) \text{ and } y \in g(x)\}; \tag{22}$$

b) $h : U \to P^*(S^{(n+p)}), U \in P^*(S^{(m)})$; *the function* $\widehat{\Upsilon} : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n \times \mathbf{B}^p$ *and the systems* $f : S^{(n)} \times U \to P^*(S^{(n+p)}), g : S^{(n)} \to P^*(S^{(n)})$ *exist such that (21) is true as well as*

$$\forall (y,u) \in S^{(n)} \times U, \exists \lim_{t\to\infty} \widehat{\Upsilon}(y(t),u(t)) \Longrightarrow \Longrightarrow \forall (x,z) \in f(y,u), \lim_{t\to\infty} (x(t),z(t)) = \lim_{t\to\infty} \widehat{\Upsilon}(y(t),u(t)), \tag{23}$$

$$\forall u \in U, h(u) = \{(x,z) | \exists y \in S^{(n)}, (x,z) \in f(y,u) \text{ and } y \in g(x)\}. \tag{24}$$

The two conditions a), b) have been drawn in Figure 3.

Remark 42 *We take a look at item a) of the previous definition. A system f having the property that Υ exists with (20) true is called combinational (or race-free stable relative to the function Υ). Property (20) shows that f is a combinational system that computes the function Υ and (21) shows that g is a combinational system that computes the identity function $1_{\mathbf{B}^n}$; (22) links f and g. As g from (21) models the delay elements, we conclude that the Huffman asynchronous systems consist in combinational systems f having feedback loops with delay elements. The interpretation is similar for item b).*

Notation 43 *For* $\Upsilon : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$ *and* $\lambda \in \mathbf{B}^n$, *we denote by* $\widetilde{\Upsilon}^\lambda : \mathbf{B}^n \times \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$ *the function:* $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \nu \in \mathbf{B}^m$,

$$\widetilde{\Upsilon}^\lambda(\mu, \mu', \nu) = (\overline{\lambda_1} \cdot \mu_1 \oplus \lambda_1 \cdot \Upsilon_1(\mu', \nu), ..., \overline{\lambda_n} \cdot \mu_n \oplus \lambda_n \cdot \Upsilon_n(\mu', \nu)).$$

Theorem 44 *Let be* $d > 0$ *and the systems* $f : S^{(n)} \times S^{(m)} \to P^*(S^{(n)})$, $g : S^{(n)} \to S^{(n)}$, $h : S^{(m)} \to P^*(S^{(n)})$ *defined in the next manner:*

$$\forall (y, u) \in S^{(n)} \times S^{(m)}, f(y, u) =$$

$$= \{x | x(t) = \widetilde{\Upsilon}^{\alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...}(x(t-0), y(t-0), u(t-0)),$$

$$x(-\infty + 0) \in \mathbf{B}^n, \alpha \in \Pi_n, (t_k) \in Seq, \forall k \in \mathbf{N}, t_{k+1} - t_k > d\},$$

$$g(x) = x \circ \tau^d,$$

$$\forall u \in S^{(m)}, h(u) = \{x | x \in f(g(x), u)\}.$$

Then:

a) f *satisfies (20),*
b) g *satisfies (21),*
c) h *satisfies (22),*
d) $h \subset \Sigma^-_\Upsilon$.

Proof. a) We choose arbitrarily $\mu \in \mathbf{B}^n, \alpha \in \Pi_n, (t_k) \in Seq$ such that $\forall k \in \mathbf{N}, t_{k+1} - t_k > d$ and let be $t' \in \mathbf{R}$ with the property that

$$\forall t'' \geq t', \Upsilon(y(t''), u(t'')) = \Upsilon(y(t'), u(t')).$$

We fix some $k \in \mathbf{N}$ with $t_k > t'$. The sets $I_k, ..., I_{k'}$ are defined by

$$I_k = \{i | i \in \{1, ..., n\}, \alpha_i^k = 1\},$$

$$...$$

$$I_{k'} = \{i | i \in \{1, ..., n\}, \alpha_i^{k'} = 1\}$$

$$I_k \cup ... \cup I_{k'} = \{1, ..., n\}.$$

The existence of $I_k, ..., I_{k'}$ is assured by the fact that α is progressive. We have

$t = t_k : \forall i \in I_k$,

$$x_i(t_k) = \widetilde{\Upsilon}_i^{\alpha^k}(x(t_k - 0), y(t_k - 0), u(t_k - 0))$$
$$= \Upsilon_i(y(t_k - 0), u(t_k - 0)) = \Upsilon_i(y(t'), u(t')),$$

$t \in (t_k, t_{k+1}) : \forall i \in I_k$,

$$x_i(t) = \widetilde{\Upsilon}_i^0(x(t-0), y(t-0), u(t-0)) = x_i(t-0) = \Upsilon_i(y(t'), u(t')),$$

$t = t_{k+1} : \forall i \in I_k \cup I_{k+1}$,

$$x_i(t_{k+1}) = \widetilde{\Upsilon}_i^{\alpha^{k+1}}(x(t_{k+1}-0), y(t_{k+1}-0), u(t_{k+1}-0))$$
$$= \begin{cases} x_i(t_{k+1}-0), i \in I_k \setminus I_{k+1} \\ \Upsilon_i(y(t_{k+1}-0), u(t_{k+1}-0)), i \in I_{k+1} \end{cases} = \Upsilon_i(y(t'), u(t')),$$

$t \in (t_{k+1}, t_{k+2}) : \forall i \in I_k \cup I_{k+1}$,

$$x_i(t) = \widetilde{\Upsilon}_i^0(x(t-0), y(t-0), u(t-0)) = x_i(t-0) = \Upsilon_i(y(t'), u(t')),$$

...

$t = t_{k'} : \forall i \in I_k \cup ... \cup I_{k'}$,

$$x_i(t_{k'}) = \widetilde{\Upsilon}_i^{\alpha^{k'}}(x(t_{k'}-0), y(t_{k'}-0), u(t_{k'}-0))$$
$$= \begin{cases} x_i(t_{k'}-0), i \in (I_k \cup ... \cup I_{k'-1}) \setminus I_{k'} \\ \Upsilon_i(y(t_{k'}-0), u(t_{k'}-0)), i \in I_{k'} \end{cases} = \Upsilon_i(y(t'), u(t')).$$

We have obtained that

$$x(t_{k'}) = \Upsilon(y(t'), u(t')).$$

Furthermore:

$t \in (t_{k'}, t_{k'+1}) : \forall i \in \{1, ..., n\}$,

$$x_i(t) = \widetilde{\Upsilon}_i^0(x(t-0), y(t-0), u(t-0)) = x_i(t-0) = \Upsilon_i(y(t'), u(t')),$$

$t = t_{k'+1} : \forall i \in \{1, ..., n\}$,

$$x_i(t_{k'+1}) = \widetilde{\Upsilon}_i^{\alpha^{k'+1}}(x(t_{k'+1}-0), y(t_{k'+1}-0), u(t_{k'+1}-0))$$
$$= \begin{cases} x_i(t_{k'+1}-0), \alpha_i^{k'+1} = 0 \\ \Upsilon_i(y(t_{k'+1}-0), u(t_{k'+1}-0)), \alpha_i^{k'+1} = 1 \end{cases} = \Upsilon_i(y(t'), u(t'))$$

and we can prove by induction on k'' that

$$\forall k'' \geq k', \forall t \in (t_{k''}, t_{k''+1}), x(t) = \Upsilon(y(t'), u(t')),$$

$$\forall k'' \geq k', x(t_{k''}) = \Upsilon(y(t'), u(t'))$$

thus

$$\forall t'' \geq t_{k'}, x(t'') = \Upsilon(y(t'), u(t')).$$

b) Some $t' \in \mathbf{R}$ exists so that $\forall t'' \geq t', x(t'') = x(t')$, wherefrom

$$\forall t'' \geq t' + d, y(t'') = g(x)(t'') = x(t'' - d) = x(t').$$

c) Obvious.

d) Let $u \in S^{(m)}$ and $x \in h(u)$ be arbitrary, fixed, in other words $\mu \in \mathbf{B}^n, \alpha \in \Pi_n$ and $(t_k) \in Seq$ exist such that $\forall k \in \mathbf{N}, t_{k+1} - t_k > d$ and

$$\begin{cases} x(-\infty + 0) = \mu \\ \forall i \in \{1, ..., n\}, x_i(t) = \begin{cases} \Upsilon_i(x(t_k - d - 0), u(t_k - 0)), \\ \quad \exists k \in \mathbf{N}, t = t_k \text{ and } \alpha_i^k = 1 \\ x_i(t - 0), \text{ otherwise} \end{cases} \end{cases}. \tag{25}$$

We take into account the fact that $\forall k \geq 1, t_k - d > t_{k-1}$ and we infer that

$$\forall k \in \mathbf{N}, x(t_k - d - 0) = x(t_k - 0).$$

With the notation

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

equation (25) is equivalent with (i.e. it has the same solution like)

$$\begin{cases} x(-\infty + 0) = \mu \\ x(t) = \Upsilon^{\rho(t)}(x(t - 0), u(t - 0)) \end{cases},$$

thus (see Theorem 33), $x(t) = \Upsilon^{-\rho}(t, \mu, u)$. ∎

Remark 45 *Another version of Theorem 44 is obtained if we want to get at d) $h \subset \Sigma_\Upsilon$ and two other versions of this theorem result by using Definition 41 b) (instead of Definition 41 a)).*

9 Conclusions

Remark 46 *The asynchronous systems theory can be made without requests of regularity and this leaves open the possibility that certain systems (i.e. the systems that have no generator function) cannot be implemented.*

Example 47 *The system $f : U \to S, U \in P^*(S)$,*

$$U = \{0, 1\} \text{ (the two constant } \mathbf{R} \to \mathbf{B} \text{ functions)}$$

$$\forall u \in U, f(u) = \begin{cases} 0, & \text{if } u = 0 \\ \chi_{[0,1)}, & \text{if } u = 1 \end{cases}$$

has no generator function, because the candidate generator functions $\Upsilon : \mathbf{B} \times \mathbf{B} \to \mathbf{B}$ fulfill, all of them, the property that $\Sigma_\Upsilon^- = \Sigma_\Upsilon$ and

$$\forall u \in U, \forall x \in \Sigma_\Upsilon^-(u), \ x \text{ is monotonous.}$$

The inclusion $f \subset \Sigma_\Upsilon^-$ is false since $\chi_{[0,1)}$ is not a monotonous function.

Remark 48 *We conclude that the previous system f models no circuit. From this point of view, regularity is as important as nonanticipation, indicating the systems that cannot be implemented.*

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