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Morphisms and antimorphisms of Boolean evolution and antievolution functions

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Abstract

The Boolean evolution and antievolution functions model the asynchronous circuits from electronics. Our purpose is to introduce their morphisms and antimorphisms.

Keywords: Boolean function, morphism, antimorphism, evolution function, antievolution function

We denote $\mathbf{B}=\{0,1\}$ the binary Boole algebra and $\mathbf{N}_{\_}=\{-1,0,1,...\}$. Let $\Phi,\Psi,h,h':\mathbf{B}^n\rightarrow\mathbf{B}^n$, for which we define $\forall i\in\{1,...,n\},\forall\nu\in\mathbf{B}^n,\forall\mu\in\mathbf{B}^n,\Phi_i^{\nu}(\mu)=\begin{cases}\mu_i, if\ \nu_i=0,\\ \Phi_i(\mu), if\ \nu_i=1\end{cases}$. If $\forall\nu\in\mathbf{B}^n,\forall\mu\in\mathbf{B}^n$, $h(\Phi^{\nu}(\mu))=\Psi^{h'(\nu)}(h(\mu))$, we say that the **morphism** (h,h') is defined, from Φ to Ψ and if $\forall\nu\in\mathbf{B}^n,\forall\mu\in\mathbf{B}^n$, $h(\mu)=\Psi^{h'(\nu)}(h(\Phi^{\nu}(\mu)))$, we say that the **antimorphism** $(h,h')^{\sim}$ is defined, from Φ to Ψ. The sets of the morphisms and of the antimorphisms from Φ to Ψ are denoted with $Hom(\Phi,\Psi),Hom^{\sim}(\Phi,\Psi)$. We denote $\widehat{\Pi}_n=\{\alpha|\alpha:\mathbf{N}\longrightarrow\mathbf{B}^n,\forall i\in\{1,...,n\},\{k|k\in\mathbf{N},\alpha_i^k:=\alpha_i(k)=1\}$ is infinite$\}$. The functions $\widehat{\Phi},\widehat{\Phi}^{\sim}$ given by $\mathbf{B}^n\times\mathbf{N}_{\_}\times\widehat{\Pi}_n\ni(\mu,k,\alpha)\longmapsto\widehat{\Phi}^{\alpha}(\mu,k),\widehat{\Phi}^{\sim\alpha}(\mu,k)\in\mathbf{B}^n$, $\widehat{\Phi}^{\alpha}(\mu,k)=\begin{cases}\mu,\ if\ k=-1,\\ \Phi^{\alpha^0}(\mu),\ if\ k=0,\\ (\Phi^{\alpha^k}\circ\Phi^{\alpha^{k-1}}\circ...\circ\Phi^{\alpha^0})(\mu),\ if\ k\geq 1\end{cases}$, $\widehat{\Phi}^{\sim\alpha}(\mu,k)=\begin{cases}\mu,\ if\ k=-1,\\ \Phi^{\alpha^0}(\mu),\ if\ k=0,\\ (\Phi^{\alpha^0}\circ\Phi^{\alpha^1}\circ...\circ\Phi^{\alpha^k})(\mu),\ if\ k\geq 1\end{cases}$ are called **evolution** and **antievolution function** and they model the asynchronous circuits, respectively the time reversed asynchronous circuits. We have by definition the **orbit** $\widehat{Or}_{\Phi}^{\alpha}(\mu) = \{\widehat{\Phi}^{\alpha}(\mu,k)|k \in \mathbf{N}_{\_}\}$ and the **omega limit set** $\widehat{\omega}_{\Phi}^{\alpha}(\mu) = \{\lambda|\lambda \in \mathbf{B}^n, \{k|k \in \mathbf{N}_{\_}, \widehat{\Phi}^{\alpha}(\mu,k) = \lambda\}$ is infinite$\}$ and similarly for $\widehat{\Phi}^{\sim}$. For $\alpha : \mathbf{N} \longrightarrow \mathbf{B}^n$ we also define $\widehat{h'}(\alpha) : \mathbf{N} \longrightarrow \mathbf{B}^n$ by $\forall k \in \mathbf{N}, \widehat{h'}(\alpha)^k = h'(\alpha^k)$ and $\Omega_n = \{h'|\widehat{h'}(\widehat{\Pi}_n) \subset \widehat{\Pi}_n\}$. Our purpose is to introduce the morphisms and the antimorphisms of evolution and antievolution functions.

Definition 1. We consider the functions $\Phi, \Psi, h, h' : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ and we suppose that $h' \in \Omega_n$. We say that the couple (h,h') is a **morphism from** the evolution function $\widehat{\Phi}$ **to** the evolution function $\widehat{\Psi}$, denoted by $(h,h') : \widehat{\Phi} \longrightarrow \widehat{\Psi}$, if $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}_{\_}, \forall \alpha \in \widehat{\Pi}_n$, $h(\widehat{\Phi}^{\alpha}(\mu,k)) = \widehat{\Psi}^{\widehat{h'}(\alpha)}(h(\mu),k)$; (h,h') is a **morphism from** the antievolution function $\widehat{\Phi}^{\sim}$ **to** the antievolution function $\widehat{\Psi}^{\sim}$, denoted by $(h,h') : \widehat{\Phi}^{\sim} \longrightarrow \widehat{\Psi}^{\sim}$, if $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}_{\_}, \forall \alpha \in \widehat{\Pi}_n$, $h(\widehat{\Phi}^{\sim\alpha}(\mu,k)) = \widehat{\Psi}^{\sim\widehat{h'}(\alpha)}(h(\mu),k)$. We denote with $Hom(\widehat{\Phi},\widehat{\Psi})$, $Hom(\widehat{\Phi}^{\sim},\widehat{\Psi}^{\sim})$ the previous sets of morphisms.

Theorem 1. We get $Hom(\widehat{\Phi},\widehat{\Psi}) = Hom(\widehat{\Phi}^{\sim},\widehat{\Psi}^{\sim}) = \{(h,h')|(h,h') \in Hom(\Phi,\Psi)$ and $h' \in \Omega_n\}$.

Theorem 2. For $\Gamma : \mathbf{B}^n \to \mathbf{B}^n$, we have $(h,h') \in Hom(\widehat{\Phi},\widehat{\Psi}), (g,g') \in Hom(\widehat{\Psi},\widehat{\Gamma}) \Longrightarrow (g \circ h, g' \circ h') \in Hom(\widehat{\Phi},\widehat{\Gamma})$ and $(h,h') \in Hom(\widehat{\Phi}^{\sim},\widehat{\Psi}^{\sim})$, $(g,g') \in Hom(\widehat{\Psi}^{\sim},\widehat{\Gamma}^{\sim}) \Longrightarrow (g \circ h, g' \circ h') \in Hom(\widehat{\Phi}^{\sim},\widehat{\Gamma}^{\sim})$.

Definition 2. The morphism $(g \circ h, g' \circ h')$ is by definition the **composition** of (g,g') and (h,h') and its notation is $(g,g') \circ (h,h')$.

Theorem 3. Let $(h,h') \in Hom(\widehat{\Phi},\widehat{\Psi})$, $(g,g') \in Hom(\widehat{\Phi}^{\sim},\widehat{\Psi}^{\sim})$. Then $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \widehat{\Pi}_n$, $h(\widehat{Or}_{\Phi}^{\alpha}(\mu)) = \widehat{Or}_{\Psi}^{\widehat{h'}(\alpha)}(h(\mu)), g(\widehat{Or}_{\Phi}^{\sim\alpha}(\mu)) = \widehat{Or}_{\Psi}^{\sim\widehat{g'}(\alpha)}(g(\mu)), h(\widehat{\omega}_{\Phi}^{\alpha}(\mu)) = \widehat{\omega}_{\Psi}^{\widehat{h'}(\alpha)}(h(\mu)), g(\widehat{\omega}_{\Phi}^{\sim\alpha}(\mu)) = \widehat{\omega}_{\Psi}^{\sim\widehat{g'}(\alpha)}(g(\mu))$.

Theorem 4. For any $\mu \in \mathbf{B}^n$ and any $\alpha \in \widehat{\Pi}_n$, if $\widehat{\Phi}^{\alpha}(\mu,\cdot)$ is periodic, with the period $p \geq 1 : \forall k \in \mathbf{N}_{\_}, \widehat{\Phi}^{\alpha}(\mu,k) = \widehat{\Phi}^{\alpha}(\mu,k+p)$ and $(h,h') \in Hom(\widehat{\Phi},\widehat{\Psi})$, then $\widehat{\Psi}^{\widehat{h'}(\alpha)}(h(\mu),\cdot)$ is periodic with the period p; if we suppose that $\widehat{\Phi}^{\sim\alpha}(\mu,\cdot)$ is periodic, with the period $p \geq 1 : \forall k \in \mathbf{N}_{\_}, \widehat{\Phi}^{\sim\alpha}(\mu,k) = \widehat{\Phi}^{\sim\alpha}(\mu,k+p)$ and $(h,h') \in Hom(\widehat{\Phi}^{\sim},\widehat{\Psi}^{\sim})$, then $\widehat{\Psi}^{\sim\widehat{h'}(\alpha)}(h(\mu),\cdot)$ is periodic with the period p.

Theorem 5. Let $\mu \in \mathbf{B}^n$ and we suppose that $\Phi(\mu) = \mu$. If $(h, h') \in Hom(\widehat{\Phi}, \widehat{\Psi})$, then $\Psi(h(\mu)) = h(\mu)$ and $\forall \alpha \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_{\_}, \widehat{\Psi}^{\widehat{h'(\alpha)}}(h(\mu), k) = h(\mu)$; if $(h, h') \in Hom(\widehat{\Phi}^{\sim}, \widehat{\Psi}^{\sim})$, then $\Psi(h(\mu)) = h(\mu)$ and $\forall \alpha \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_{\_}, \widehat{\Psi}^{\sim\widehat{h'(\alpha)}}(h(\mu), k) = h(\mu)$.

Definition 3. We ask that $h' \in \Omega_n$. We say that the couple (h, h') is an **antimorphism from $\widehat{\Phi}^{\sim}$ to $\widehat{\Psi}$**, denoted $(h, h')^{\sim} : \widehat{\Phi}^{\sim} \longrightarrow \widehat{\Psi}$ or simply $(h, h')^{\sim}$, if $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}_{\_}, \forall \alpha \in \widehat{\Pi}_n, h(\mu) = \widehat{\Psi}^{\widehat{h'(\alpha)}}(h(\widehat{\Phi}^{\sim\alpha}(\mu, k)), k)$ and (h, h') is by definition an **antimorphism from $\widehat{\Phi}$ to $\widehat{\Psi}^{\sim}$**, denoted $(h, h')^{\sim} : \widehat{\Phi} \longrightarrow \widehat{\Psi}^{\sim}$ or $(h, h')^{\sim}$, if $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}_{\_}, \forall \alpha \in \widehat{\Pi}_n, h(\mu) = \widehat{\Psi}^{\sim\widehat{h'(\alpha)}}(h(\widehat{\Phi}^{\alpha}(\mu, k)), k)$. We use the notation $Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi})$, $Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}^{\sim})$ for the previous sets of antimorphisms.

Theorem 6. We get $Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi}) = Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}^{\sim}) = \{(h, h')^{\sim} | (h, h')^{\sim} \in Hom^{\sim}(\Phi, \Psi) \text{ and } h' \in \Omega_n\}$.

Theorem 7. a) $(h, h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi}), (g, g')^{\sim} \in Hom^{\sim}(\widehat{\Psi}, \widehat{\Gamma}^{\sim}) \Longrightarrow (g \circ h, g' \circ h') \in Hom(\widehat{\Phi}^{\sim}, \widehat{\Gamma}^{\sim})$, b) $(h, h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}^{\sim}), (g, g')^{\sim} \in Hom^{\sim}(\widehat{\Psi}^{\sim}, \widehat{\Gamma}) \Longrightarrow (g \circ h, g' \circ h') \in Hom(\widehat{\Phi}, \widehat{\Gamma})$, c) $(h, h') \in Hom(\widehat{\Phi}, \widehat{\Psi})$, $(g, g')^{\sim} \in Hom^{\sim}(\widehat{\Psi}, \widehat{\Gamma}^{\sim}) \Longrightarrow (g \circ h, g' \circ h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}, \widehat{\Gamma}^{\sim})$, d) $(h, h') \in Hom(\widehat{\Phi}^{\sim}, \widehat{\Psi}^{\sim}), (g, g')^{\sim} \in Hom(\widehat{\Psi}^{\sim}, \widehat{\Gamma}) \Longrightarrow (g \circ h, g' \circ h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Gamma})$, e) $(h, h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi}), (g, g') \in Hom(\widehat{\Psi}, \widehat{\Gamma}) \Longrightarrow (g \circ h, g' \circ h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Gamma})$, f) $(h, h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}^{\sim}), (g, g') \in Hom(\widehat{\Psi}^{\sim}, \widehat{\Gamma}^{\sim}) \Longrightarrow (g \circ h, g' \circ h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}, \widehat{\Gamma}^{\sim})$ hold.

Definition 4. In a), b) the morphism $(g \circ h, g' \circ h')$ is by definition the **composition** of the antimorphisms $(g, g')^{\sim}$ and $(h, h')^{\sim}$ and its notation is $(g, g')^{\sim} \circ (h, h')^{\sim}$. In c), d) the antimorphism $(g \circ h, g' \circ h')^{\sim}$ is by definition the **composition** of the antimorphism $(g, g')^{\sim}$ with the morphism (h, h') and it has the notation $(g, g')^{\sim} \circ (h, h')$. Similarly for $(g \circ h, g' \circ h')^{\sim}$ denoted $(g, g') \circ (h, h')^{\sim}$ from e), f).

Theorem 8. Let the functions $\Phi, \Psi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ and the antimorphisms $(h, h')^{\sim} \in Hom(\widehat{\Phi}^{\sim}, \widehat{\Psi}), (g, g')^{\sim} \in Hom(\widehat{\Phi}, \widehat{\Psi}^{\sim})$; $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \widehat{\Pi}_n$, we have $\forall \nu \in \widehat{Or}_{\Phi}^{\sim\alpha}(\mu), h(\mu) \in \widehat{Or}_{\Psi}^{\widehat{h'(\alpha)}}(h(\nu)), \forall \nu \in \widehat{Or}_{\Phi}^{\alpha}(\mu), g(\mu) \in$

$\widehat{Or}_{\Psi}^{\sim\widehat{h'}(\alpha)}(g(\nu)), \forall\nu \in \widehat{\omega}_{\Phi}^{\sim\alpha}(\mu), h(\mu) \in \widehat{\omega}_{\Psi}^{\widehat{h'}(\alpha)}(h(\nu)), \forall\nu \in \widehat{\omega}_{\Phi}^{\alpha}(\mu), g(\mu) \in \widehat{\omega}_{\Psi}^{\sim\widehat{h'}(\alpha)}(g(\nu))$.

Theorem 9. Let $\mu \in \mathbf{B}^n$ with $\Phi(\mu) = \mu$. If $(h, h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi})$, then $\Psi(h(\mu)) = h(\mu)$ and $\forall\alpha \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_{\_}, \widehat{\Psi}^{\widehat{h'}(\alpha)}(h(\mu), k) = h(\mu)$ hold; if $(h, h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}^{\sim})$, then $\Psi(h(\mu)) = h(\mu)$ and $\forall\alpha \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_{\_}, \widehat{\Psi}^{\sim\widehat{h'}(\alpha)}(h(\mu), k) = h(\mu)$ are true.

Remark 1. The next sets $Hom(\widehat{\Phi}^{\sim}, \widehat{\Psi}), Hom(\widehat{\Phi}, \widehat{\Psi}^{\sim})$, $Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi})$, $Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi}^{\sim})$ are defined like in Definition 1 and Definition 3. We can prove that $Hom(\widehat{\Phi}^{\sim}, \widehat{\Psi}) = Hom(\widehat{\Phi}, \widehat{\Psi}^{\sim}), Hom(\widehat{\Phi}^{\sim}, \widehat{\Psi}) \subset Hom(\Phi, \Psi), Hom(\widehat{\Phi}^{\sim}, \widehat{\Psi}) \subset Hom(\widehat{\Phi}, \widehat{\Psi}), Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}) = Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi}^{\sim}), Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}) \subset Hom^{\sim}(\Phi, \Psi), Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}) \subset Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi})$. These morphisms and antimorphisms are not induced by morphisms $(h, h') \in Hom(\Phi, \Psi)$ and antimorphisms $(h, h')^{\sim} \in Hom^{\sim}(\Phi, \Psi)$, i.e. theorems like 1 and 6 are false.

At the same time we notice, as a conclusion, in which manner the morphisms and the antimorphisms keep the orbits, the omega limit sets, periodicity and the fixed points.

References

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