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TOPICS IN BINARY VALUED ANALYSIS

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Abstract The study of the $\boldsymbol{R} \to \{0,1\}$ functions is sketched.

1. The Equations of the Asynchronous Automata

1.1 $\boldsymbol{B}_2 = \{0,1\}$ is the Binary Boole algebra, together with the discrete topology. $'\oplus'$ is the modulo 2 sum, $'\cdot'$ is the intersection and $\bigcup_{i\in I} a_i$ is the reunion of $a_i \in \boldsymbol{B}_2, i \in I$.

1.2 Let $x : \boldsymbol{R} \to \boldsymbol{B}_2$. If there are true

i) $\forall t \in \boldsymbol{R}, \exists \varpi \in \boldsymbol{B}_2, \exists \varepsilon \in \boldsymbol{R}, \varepsilon > 0, \forall \xi \in (t-\varepsilon, t), x(\xi) = \varpi$

ii) $\forall t \in \boldsymbol{R}, \exists \varpi^* \in \boldsymbol{B}_2, \exists \varepsilon \in \boldsymbol{R}, \varepsilon > 0, \forall \xi \in (t, t+\varepsilon), x(\xi) = \varpi^*$

then x is said to have *left limits*, respectively *right limits*. The numbers

$$\varpi \overset{not}{=} \lim_{\substack{\xi \to t \\ \xi < t}} x(\xi) = x(t-0), \ \varpi^* \overset{not}{=} \lim_{\substack{\xi \to t \\ \xi > t}} x(\xi) = x(t+0)$$

are the left and the right limits of x in t.

1.3 The functions

$$Dx(t) = x(t-0) \oplus x(t), \ D^* x(t) = x(t+0) \oplus x(t)$$

are the *left derivative* and the *right derivative* of x in t.

There are remarkable the properties:

$$DDx(t) = Dx(t), \ D^* D^* x(t) = D^* x(t)$$

$$D^* Dx(t) = Dx(t), \ DD^* x(t) = D^* x(t)$$

1.4 We define the set of the *realizable functions*

$$Real^{(n)} = \{x \mid x = (x_1, ..., x_n) : \boldsymbol{R} \to \boldsymbol{B}_2^n, x_i \text{ has left limits and right limits,}$$
$$D^* x_i(t) = 0, t \in \boldsymbol{R} \text{ and } x_i(t) = 0, t < 0 \text{ for all } i = \overline{1,n}\}$$

1.5 The *equations of the asynchronous automata* are

$$Dx_i(t) = (x_i(t-0) \oplus f_i(x(t-0), u(t-0))) \cdot$$
$$\cdot (\bigcup_{\xi \in (t-\tau_i, t)} Df_i(x(\xi), u(\xi)) \oplus 1) \cdot \eta(t-\tau_i) \oplus x_i^0 \cdot \delta(t)$$

$i = \overline{1,n}$, where $\eta(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases}$, $\delta(t) = \begin{cases} 1, t = 0 \\ 0, t \neq 0 \end{cases}$ and

- $x \in Real^{(n)}$ is the *state function*; $x^0 \in \boldsymbol{B}_2^n$ is the *initial state*
- $u \in Real^{(m)}$ is the *input* (the *control*) *function*

- $f : \boldsymbol{B}_2^n \times \boldsymbol{B}_2^m \to \boldsymbol{B}_2^n$ is the *generator function*
- $\tau_i > 0, i = \overline{1, n}$ are the *delays*, or the *switching time parameters*

1.6 With the notations from fig 1.6.1 a, the circuits that the equations 1.5 model are of the sort from fig 1.6.1 b (so called *switching circuits*).

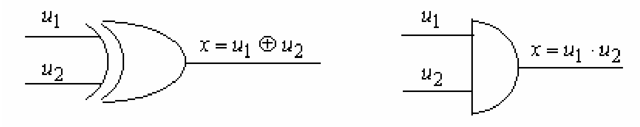

fig 1.6.1 a

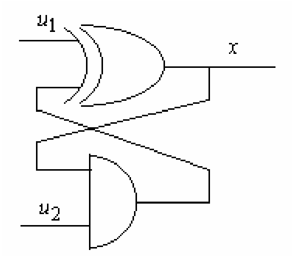

fig 1.6.1 b

The solution of 1.5 always exists and it is unique.

2. Further Developments

Writing the equations of the asynchronous automata supposed some analogies with the mathematics of the $\boldsymbol{R} \to \boldsymbol{R}$ functions, that can be continued. We shall give here just a hint on these constructions.

2.1 The notion of left (right) limit of $x : \boldsymbol{R} \to \boldsymbol{B}_2$ may be generalized to: *superior left limit*

$$\overline{x(t-0)} = \limsup_{\substack{\xi \to t \\ \xi < t}} x(\xi) = \lim_{\substack{t_1 \to t \\ t_1 < t}} \bigcup_{\xi \in (t_1, t)} x(\xi)$$

and *inferior left limit*, *limits relative to a subset* $A \subset \boldsymbol{R}$; we have also *left* (and *right*) *limits of the subsets* $A \subset \boldsymbol{R}$ defined by

$$A_- = \{t \mid t \in \boldsymbol{R} \vee \{\infty\}, \chi_A(t-0) = 1\}$$

where $\chi_{(\cdot)}$ is the characteristic function. All these limits are related to *left* (*right*) *continuities* and *derivatives*.

2.2 Let $X \neq \emptyset$ and $R \subset \{A \mid A \subset X\}$ closed relative to the symmetrical difference Δ and the intersection $\wedge$ (= ring of subsets of X). The function $\mu : R \to \boldsymbol{B}_2$ is called *Boolean measure* if for any sequence of sets $A_n \in R, n \in \boldsymbol{N}$ disjoint two by two with $\bigvee_{n \in \boldsymbol{N}} A_n \in R$, we have that the set $\{n \mid n \in \boldsymbol{N}, \mu(A_n) = 1\}$ is finite and it is true

$$\mu(\bigvee_{n \in \boldsymbol{N}} A_n) = \mu(A_0) \oplus \mu(A_1) \oplus \ldots$$

2.3 Let us consider the function $x : X \to \boldsymbol{B}_2$. The number

$$\int_A x D\mu = \mu(A \wedge \{t \mid t \in X, x(t) = 1\})$$

- where $A \subset X$ and $A \wedge \{t \mid t \in X, x(t) = 1\} \in R$ - is the *integral of* x *on* A *relative to* μ.

We have also *integrals in the Stieltjes* and *Riemann sense* and *primitives*. For $\gamma : [a,b] \to \boldsymbol{B}_2^2$ subject to some conditions - called *path* and ω a *Boolean form* on $\boldsymbol{B}_2^2$ (not defined in this paper), the *integral* $\int_\gamma \omega$ may be defined.

2.4 The above facts give the possibility of defining *convolution products* and *distributions* over $\boldsymbol{R} \to \boldsymbol{B}_2^n$ test functions. It is called *test function in the general sense* a function $x : \boldsymbol{R} \to \boldsymbol{B}_2^n$ with the property that x_i have left limits and right limits, $i = \overline{1,n}$.

It is called *test function in the restricted sense*, a test function in the general sense, with the property that the set $\{t \mid x(t) \neq 0\}$ is bounded.

Two sets of test functions are consequently defined and let K_n be any of them. For $x \in K_n$ and $x_\tau(t) \overset{def}{=} x(t - \tau)$, it is called *distribution over* K_n a function $f : K_n \to \boldsymbol{B}_2$ fulfilling the next requirements:

i) $\forall x, y \in K_n, f(x \oplus y) = f(x) \oplus f(y)$

ii) $\forall x \in K_n, g(\tau) = f(\psi \cdot x_\tau)$ has left limits and right limits for all $\psi : \boldsymbol{R} \to \boldsymbol{B}_2$ with left limits and right limits.

The set of the distributions is noted with $K_n^{'}$.

3. Differentials

3.1 For the function $f : \boldsymbol{R} \to \boldsymbol{B}_2$ and $a \in \boldsymbol{R}$, the next statements are equivalent:

i) $f(a-0), f(a+0)$ exist (or $Df(a), D^* f(a)$ exist)

ii) the constants $a_1, a_2 \in \boldsymbol{B}_2$ and the function $\omega : \boldsymbol{R} \to \boldsymbol{B}_2$ exist with the property that ω is null in some neighborhood $(a - \varepsilon, a + \varepsilon)$ of a (i.e. $\lim_{x \to a} \omega(x) = \omega(a) = 0$) so that, with the notations $\overset{*}{\eta}_+(x) = \begin{cases} 1, x < 0 \\ 0, x \geq 0 \end{cases}, \eta_-(x) = \begin{cases} 1, x > 0 \\ 0, x \leq 0 \end{cases}$ we have:

$$f(x) = f(a) \oplus a_1 \cdot \overset{*}{\eta}_+(x-a) \oplus a_2 \cdot \eta_-(x-a) \oplus \omega(x)$$

It may be proved that if ii) is true, then $Df(a)$ and $D^* f(a)$ exist and

$$a_1 = Df(a), a_2 = D^* f(a)$$

3.2 In any of the above situations i), ii) f is said to be *differentiable in* a and the function

$$df(a)(x) = Df(a) \cdot \overset{*}{\eta}_+(x-a) \oplus D^* f(a) \cdot \eta_-(x-a)$$

is called the *differential of* f *in* a. It is true:

$$d(df(a))(a) = df(a)$$

3.3 Let us consider the function $f : \boldsymbol{R}^2 \to \boldsymbol{B}_2, \boldsymbol{R}^2 \ni (x,y) \mapsto f(x,y) \in \boldsymbol{B}_2$ and the point $(a,b) \in \boldsymbol{R}^2$. By fixing one of x, y to a, b, limits of the sort $f(a-0,b), f(a,b+0),...$ may be obtained, together with the corresponding derivatives:

$$D_x f(a,b) \overset{def}{=} f(a-0,b) \oplus f(a,b)$$

etc. The number $\varpi \in \boldsymbol{B}_2$ defined by

$$\exists \varepsilon_1 > 0, \forall \xi_1 \in (a-\varepsilon_1, a), \exists \varepsilon_2 > 0, \forall \xi_2 \in (b, b+\varepsilon_2), f(\xi_1, \xi_2) = \varpi$$

is usually written

$$v = \lim_{\substack{x_1 \to a \\ x_1 < a}} \lim_{\substack{x_2 \to b \\ x_2 > b}} f(x_1, x_2) = f(a \underset{2}{-} 0, b \underset{1}{+} 0)$$

where the order of taking limits was indicated by the numbers 1,2. If this order is not essential, we shall simply write $f(a-0, b+0)$.

3.4 The function $f : \boldsymbol{R}^2 \to \boldsymbol{B}_2, f = f(x,y)$ is *differentiable in* $(a,b) \in \boldsymbol{R}^2$ if any of the following equivalent conditions is fulfilled:

i) $f(a-0,b), f(a+0,b), f(a,b-0), f(a,b+0), f(a-0,b-0), f(a-0,b+0),$ $f(a+0,b-0), f(a+0,b+0)$ exist (or $D_x f(a,b), D_x^* f(a,b), D_y f(a,b), D_y^* f(a,b),$ $D_x D_y f(a,b) = D_y f(a-0,b) \oplus D_y f(a,b), D_x^* D_y f(a,b), D_x D_y^* f(a,b), D_x^* D_y^* f(a,b)$ exist).

This means that in a neighborhood $\{(x,y) \mid (x-a)^2 + (y-b)^2 < \varepsilon\}$ of (a,b) the form of f is like in fig 3.4.1

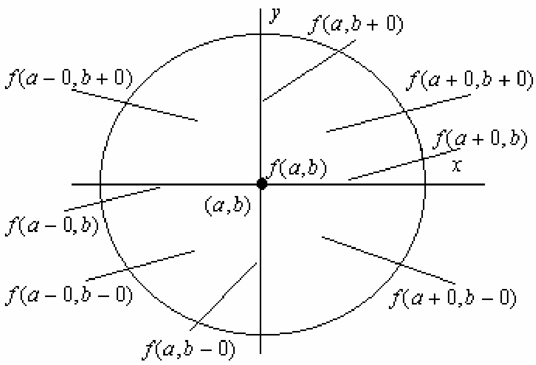

fig 3.4.1

ii) the constants $a_1, ..., a_8 \in \boldsymbol{B}_2$ and the function $\omega : \boldsymbol{R}^2 \to \boldsymbol{B}_2$ exist with the property that ω is null in some neighborhood $\{(x,y) \mid (x-a)^2 + (y-b)^2 < \varepsilon\}$ of (a,b) (i.e. $\lim\limits_{(x,y)\to(a,b)} \omega(x,y) = \omega(a,b) = 0$) so that

$$\begin{aligned} f(x,y) = {} & f(a,b) \oplus a_1 \cdot \eta_+^*(x-a) \oplus a_2 \cdot \eta_-(x-a) \oplus \\ & \oplus a_3 \cdot \eta_+^*(y-b) \oplus a_4 \cdot \eta_-(y-b) \oplus \\ & \oplus a_5 \cdot \eta_+^*(x-a) \cdot \eta_+^*(y-b) \oplus a_6 \cdot \eta_+^*(x-a) \cdot \eta_-(y-b) \oplus \\ & \oplus a_7 \cdot \eta_-(x-a) \cdot \eta_+^*(y-b) \oplus a_8 \cdot \eta_-(x-a) \cdot \eta_-(y-b) \oplus \omega(x,y) \end{aligned}$$

It may be proved that if ii) is true, then the derivatives from i) exist, the order of taking the second order derivatives is not essential and

$$a_1 = D_x f(a,b), a_2 = D_x^* f(a,b), a_3 = D_y f(a,b), a_4 = D_y^* f(a,b)$$

$$a_5 = D_x D_y f(a,b),\, a_6 = D_x D_y^* f(a,b),\, a_7 = D_x^* D_y f(a,b),\, a_8 = D_x^* D_y^* f(a,b)$$

3.5 Let $f : \boldsymbol{R}^2 \to \boldsymbol{B}_2$ be differentiable in (a,b). The function

$$\begin{aligned} df(a,b)(x,y) = {} & D_x f(a,b) \cdot \eta_+^*(x-a) \oplus D_x^* f(a,b) \cdot \eta_-(x-a) \oplus \\ & \oplus D_y f(a,b) \cdot \eta_+^*(y-b) \oplus D_y^* f(a,b) \cdot \eta_-(y-b) \oplus \\ & \oplus D_x D_y f(a,b) \cdot \eta_+^*(x-a) \cdot \eta_+^*(y-b) \oplus D_x D_y^* f(a,b) \cdot \eta_+^*(x-a) \cdot \eta_-(y-b) \oplus \\ & \oplus D_x^* D_y f(a,b) \cdot \eta_-(x-a) \cdot \eta_+^*(y-b) \oplus D_x^* D_y^* f(a,b) \cdot \eta_-(x-a) \cdot \eta_-(y-b) \end{aligned}$$

is called the *differential of* f *in* (a,b). The next property is remarkable:

$$d(df(a,b))(a,b) = df(a,b)$$

4. A Look towards Temporal Logic

4.1 Let $x : \boldsymbol{R} \to \boldsymbol{B}_2$ and X be a proposition (a formula). By replacing $x(t)$ with $R_t(X)$ - read: "it is the case that X at t"- where R_t is a unary operator appearing for example at Rescher and Urquhart, we put the problem of the derivative

$$R_t(DX) \equiv R_{t-0}(X) \underline{\vee} R_t(X)$$

$\underline{\vee}$ being the strong disjunction. It characterises the dependency on time of the statement: "the truth value of $R_t(X)$ has changed". The system characterised by 1.5 processes knowledge and its input is to be interpreted as knowledge arriving from outside the system, given by a book, a teacher or simply by the eyes or the ears.

4.2 It is nice to remark that the conditions that are generally put in the systems theory have an analogue in logic:

- stability, meaning that steady states are reached $\leftrightarrow$ for the information that is communicated to the knowledge processor, conclusions exist
- the input switches slowly enough for the system to reach the steady states (= the fundamental mode) $\leftrightarrow$ the teacher teches slowly enough so that the lesson is comprehensible, i.e. the conclusions are reached
- the technical condition of good running (also called semi-modularity), a property of invariance relative to $\tau_i, i = \overline{1,n}$ $\leftrightarrow$ the reasoning itself should not depend on the speed of processing the information.

4.5 **Remark** Rescher and Urquhart give axioms that do not make a clear distinction (most of them) between time and space. It might be interesting to think again at the facts from the paragraph 4 of the paper by replacing time with space.