

Asynchronous pseudo-systems

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Abstract

The paper introduces the concept of asynchronous pseudo-system. Its purpose is to correct/generalize/continue the study of the asynchronous systems (the models of the asynchronous circuits) that has been started in [1], [2].

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1 Introduction

The study of the asynchronous systems [1], [2] was generated by the mathematical models of the asynchronous circuits from the digital electrical engineering. What we have proposed there is that starting from the very general notion of system (non-deterministic, in the input-output sense), by the addition of definitions=axioms to rediscover one by one the properties of the models of the asynchronous circuits. Roughly speaking, the signals are the differentiable, right continuous $\mathbb{R}^+ \rightarrow \mathbb{R}^n$ functions with initial values (i.e. with limit when $t \rightarrow 0^+$) and an (asynchronous) system is a multi-valued function that associates to a signal $\mathbb{R}^+ \rightarrow \mathbb{R}^m$ called (admissible) input, a non-empty set of $\mathbb{R}^+ \rightarrow \mathbb{R}^n$ signals, called (possible) states.

The purpose of this work is that of correcting/improving/generalizing the frame of these papers and the main concept is that of pseudo-system, representing a multi-valued function from differentiable right continuous $\mathbb{R}^+ \rightarrow \mathbb{R}^m$

functions called inputs to (empty or non-empty) sets of differentiable right continuous $\mathbb{R} \rightarrow \mathbb{F}^n$ functions, called states. In other words, we have relaxed two conditions relative to the systems:

- the functions $\mathbb{R} \rightarrow \mathbb{F}^n$ without limit when $t \rightarrow \infty$ (without initial values) are accepted
- to an input $u : \mathbb{R} \rightarrow \mathbb{F}^m$ there may correspond an empty set of states, i.e. we accept the existence of non-admissible inputs.

We prefer this approach in order to underline the duality between the initial states and initial time on one hand and the final states and final time, on the other hand. Besides, we must take into account the fact that very simple circuits like the RS latch for example have non-admissible inputs ($R \wedge S = 1$).

We define and characterize the pseudo-systems, the initial and the final states, the initial and the final time, the initial and the final state functions, the pseudo-subsystems, the dual pseudo-systems, the inverse pseudo-systems, the direct product, the parallel and the serial connection, the complement, the intersection and the reunion of the pseudo-systems. The conclusions are expressed in the last section, where we define the systems as special cases of pseudo-systems whose admissible inputs and possible states are signals and we also show how the previous topics related with the pseudo-systems are particularized to the case of the systems.

We have written in full details all the dual results. The proofs are generally elementary and some of them have been omitted, some of them have been included for the reason of making the exposure as readable as possible. The dual proofs have been omitted.

2 Differentiable functions. Signals

We note with $\mathcal{B} = \mathbb{F}^n$ the Boole algebra with two elements and with $\hat{A}_A : \mathbb{R} \rightarrow \mathcal{B}$ the characteristic function of the set $A \subseteq \mathbb{R}$. The differentiable functions $x : \mathbb{R} \rightarrow \mathcal{B}^n$ are by definition of the form:

$$x(t) = \dots \odot x(t_{i-1}) \uparrow \hat{A}_{[t_{i-1}, t]}(t) \odot x\left(\frac{t_{i-1} + t_0}{2}\right) \uparrow \hat{A}_{(t_{i-1}, t_0)}(t) \odot \dots \quad (1)$$

$$\odot x(t_0) \uparrow \hat{A}_{[t_0, t]}(t) \odot x\left(\frac{t_0 + t_1}{2}\right) \uparrow \hat{A}_{(t_0, t_1)}(t) \odot x(t_1) \uparrow \hat{A}_{[t_1, t]}(t) \odot \dots$$

where $\dots < t_{i-1} < t_0 < t_1 < \dots$ is an upper and lower unbounded sequence and \mathbb{R} is the dense ($\exists t \in \mathbb{R}; \exists t^0 \in \mathbb{R}; t < t^0 \Rightarrow \exists t'' \in \mathbb{R}; t < t'' < t^0$) and linear (i.e. totally ordered: $\exists t \in \mathbb{R}; \exists t^0 \in \mathbb{R}; t \cdot t^0$ or $t^0 \cdot t$) time set. If in (1) $x(t_k) = x\left(\frac{t_k + t_{k+1}}{2}\right); k \in \mathbb{Z}$, then x is right continuous and it is of the form

$$x(t) = \dots \odot x(t_{i-1}) \uparrow \hat{A}_{[t_{i-1}, t_0)}(t) \odot x(t_0) \uparrow \hat{A}_{[t_0, t_1)}(t) \odot \dots$$

The set of the (n-dimensional) differentiable, right continuous functions x is noted with $\mathfrak{S}^{(n)}$.¹

¹The differentiable left continuous functions

$$x(t) = \dots \odot x(t_0) \uparrow \hat{A}_{(t_{i-1}, t_0]}(t) \odot x(t_1) \uparrow \hat{A}_{(t_0, t_1]}(t) \odot \dots$$

We consider the next properties of some $x \in \mathfrak{S}^{(n)}$:

$$x^1 \in \mathbb{B}^n; \exists t_0 \in \mathbb{R}; \forall t < t_0; x(t) = x^1 \quad (2)$$

$$x^{10} \in \mathbb{B}^n; \exists t_f \in \mathbb{R}; \forall t > t_f; x(t) = x^{10} \quad (3)$$

where in (2) $x^1 \in \mathbb{B}^n; \exists t_0 \in \mathbb{R}$ commute and in (3) $x^{10} \in \mathbb{B}^n; \exists t_f \in \mathbb{R}$ commute also.

If (2) is fulfilled:

- x^1 is unique and is called the initial value of x . We shall note it sometimes with $\lim_{t \rightarrow t_0^-} x(t)$, $x(t_0 - 0)$ or with $x(t_0 - 0)$

- t_0 is not unique, since any $t_0^0 < t_0$ satisfies (2) too. It is called the initial time of x .

If (3) is fulfilled:

- x^{10} is unique and is called the final value of x . The usual notations are $\lim_{t \rightarrow t_f^+} x(t)$, $x(t_f)$ and $x(t_f + 0)$

- t_f is not unique, because any $t_f^0 > t_f$ satisfies (3) too. It is called the final time of x .

We call (n_i dimensional) signal a function $x \in \mathfrak{S}^{(n)}$ with the property that (2) is satisfied. The signals are represented under the form:

$$x(t) = x(t_0 - 0) \delta_{[t_0 - 1; t_0)}(t) \otimes x(t_0) \delta_{[t_0; t_1)}(t) \otimes x(t_1) \delta_{[t_1; t_2)}(t) \otimes \dots$$

where $t_0 < t_1 < t_2 < \dots$ is unbounded. The set of the signals is noted with $\mathfrak{S}^{(n)}$.

Dually, we call (n_i dimensional) signal^a a function $x \in \mathfrak{S}^{(n)}$ with the property that (3) is true and such functions are represented under the form

$$x(t) = \dots \otimes x(t_{i-2}) \delta_{[t_{i-2}; t_{i-1})}(t) \otimes x(t_{i-1}) \delta_{[t_{i-1}; t_0)}(t) \otimes x(t_0) \delta_{[t_0; 1)}(t)$$

where $\dots < t_{i-2} < t_{i-1} < t_0$ is unbounded. The set of the signals^a is noted with $\mathfrak{S}^{(n)a}$.

We shall often write $\mathfrak{S}; \mathfrak{S}_i; \mathfrak{S}^a$ instead of $\mathfrak{S}^{(1)}; \mathfrak{S}^{(1)}; \mathfrak{S}^{(1)a}$.

We use the notations $P(L) = \{K\} \times \frac{1}{2} L; g$ and $P^a(L) = \{K\} \times \frac{1}{2} L; K \in \mathfrak{E}; g$, where L is any of $\mathbb{B}^n; \mathfrak{S}^{(n)}; \mathfrak{S}^{(n)}; \mathfrak{S}^{(n)a}$.

3 Pseudo-systems

Definition 3.1 The functions $f : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$ are called (asynchronous) pseudo-systems. The elements $u \in \mathfrak{S}^{(m)}$ are called inputs (in the pseudo-system): admissible if $f(u) \in \mathfrak{E}$; and non-admissible otherwise, while the elements $x \in f(u)$ are called (possible) states (of the pseudo-system), or (possible) outputs (from the pseudo-system).

give an equivalent manner of writing this paper. In previous works we have associated non-anticipation with right continuity and anticipation with left continuity.

Remark 3.2 The pseudo-systems are multi-valued functions that associate to each input u the set of the possible states $f(u)$, the origin of the concept being situated in the modeling of the asynchronous circuits from digital engineering.

A non-admissible input, i.e. an input for which $f(u) = \emptyset$, is considered a cause of no effect that can be expressed by f and the null pseudo-system that is defined by $\forall u \in \mathfrak{S}^{(m)}; f(u) = \emptyset$; represents the limit situation when f does not express a determination between the elements of $\mathfrak{S}^{(m)}$ and the elements of $\mathfrak{S}^{(n)}$ (f models nothing). The other limit situation is represented by the total pseudo-system, defined by $\forall u \in \mathfrak{S}^{(m)}; f(u) = \mathfrak{S}^{(n)}$ (f models every circuit with m -dimensional inputs and n -dimensional outputs); for this pseudo-system, all the inputs are admissible.

The multi-valued character of the cause-effect association is due to statistical fluctuations in the fabrication process, variations in ambient temperature, power supply etc.

In applications, the pseudo-systems are defined sometimes not explicitly, like before, but implicitly, by a system of equations and inequalities where u is given, t is the time variable, x is the unknown and the temporal logical connectors depending on them are differentiable in general (they are not right continuous).

Example 3.3 The pseudo-system $f : \mathfrak{S} \rightarrow P(\mathfrak{S})$ is defined by the double inequality

$$\forall u \in \mathfrak{S}; \forall t \in \mathbb{R}; \forall x \in \mathfrak{S} \quad \bigwedge_{\tau \in [t; t+d]} u(\tau) \cdot x(t) \cdot \bigvee_{\tau \in [t; t+d]} u(\tau) \quad (4)$$

where $d > 0$. When $u; x \in \mathfrak{S}$, the connectors $\bigwedge_{\tau \in [t; t+d]} u(\tau)$ and $\bigvee_{\tau \in [t; t+d]} u(\tau)$ are just differentiable, they are not right continuous.

4 Initial states and final states

Remark 4.1 We state the next properties of the pseudo-system f :

$$\forall u \in \mathfrak{S}^{(m)}; \forall x \in f(u); \forall \tau \in \mathbb{B}^n; \forall t_0 \in \mathbb{R}; \forall t < t_0; x(t) = \tau \quad (5)$$

$$\forall u \in \mathfrak{S}^{(m)}; \forall \tau \in \mathbb{B}^n; \forall x \in f(u); \forall t_0 \in \mathbb{R}; \forall t < t_0; x(t) = \tau \quad (6)$$

$$\forall \tau \in \mathbb{B}^n; \forall u \in \mathfrak{S}^{(m)}; \forall x \in f(u); \forall t_0 \in \mathbb{R}; \forall t < t_0; x(t) = \tau \quad (7)$$

$$\forall u \in \mathfrak{S}^{(m)}; \forall x \in f(u); \forall \tau \in \mathbb{B}^n; \forall t_f \in \mathbb{R}; \forall t > t_f; x(t) = \tau \quad (8)$$

$$\forall u \in \mathfrak{S}^{(m)}; \forall \tau \in \mathbb{B}^n; \forall x \in f(u); \forall t_f \in \mathbb{R}; \forall t > t_f; x(t) = \tau \quad (9)$$

$$\forall \tau \in \mathbb{B}^n; \forall u \in \mathfrak{S}^{(m)}; \forall x \in f(u); \forall t_f \in \mathbb{R}; \forall t > t_f; x(t) = \tau \quad (10)$$

where in (5) τ and t_0 depend on x only, thus $\forall \tau \in \mathbb{B}^n; \forall t_0 \in \mathbb{R}$ commute and similarly for τ and t_f in (8). We observe the dualities between (5) and (8); (6) and (9); (7) and (10) and on the other hand we remark the truth of the implications

$$(7) \Rightarrow (6) \Rightarrow (5)$$

$$(10) \Rightarrow (9) \Rightarrow (8)$$

If f is the null pseudo-system, it fulfills trivially all the properties (5),..., (10). If (5) is true with f non-null, it defines a partial function $\mathfrak{S}^{(n)} : B^n$ that associates to each $x \in B^n$ $f(x)$ its initial value x^1 . If (6) is true with f non-null, it defines a partial function $\mathfrak{S}^{(m)} : B^n$ that associates to each admissible input u the common initial value x^1 of all $x \in B^n$ such that $f(x, u) = x^1$. Dually, if f is non-null and (8), (9) are true, they define two partial functions $\mathfrak{S}^{(n)} : B^n$ and $\mathfrak{S}^{(m)} : B^n$.

If f is null, any $x^1 \in B^n$ makes (7) and (10) true; otherwise, the value of x^1 is uniquely defined by either of (7) and (10).

Definition 4.2 If f satisfies (5), we say that it has initial states. The vectors x^1 are called in this case (the) initial states (of f), or (the) initial values of the states (of f).

Definition 4.3 We suppose that f satisfies (6). We say in this situation that it has race-free, or delay-insensitive initial states and the initial states x^1 are called race-free, or delay-insensitive themselves.

Definition 4.4 When f satisfies (7), we use to say that it has a (constant) initial state x^1 . We say in this case that f is initialized and that x^1 is its (constant) initial state.

Definition 4.5 If f satisfies (8), it is called absolutely stable and we also say that it has final states. The vectors x^1 have in this case the name of final states (of f), or of final values of the states (of f), or of steady states (of f), or of steady values of the states (of f).

Definition 4.6 If f fulfills the property (9), it is called absolutely race-free stable, or absolutely delay-insensitive and we also say that it has race-free final states. The final states x^1 are called in this case race-free, or delay-insensitive.

Definition 4.7 We suppose that the pseudo-system f satisfies (10). Then it is called absolutely constantly stable or equivalently we say that it has a (constant) final state. The vector x^1 is called in this situation (constant) final state.

Remark 4.8 The previous terminology is related with the dualities initial-final, initialized-absolutely stable as well as with hardware engineering. In hardware engineering, 'race' means: 'which coordinate of x switches first is the winner' or perhaps 'several ways to go' and in this case 'race-free' means 'one way to go'; and delay-insensitivity means (vaguely) 'for any perturbations in the fabrication process', see Remark 3.2.

5 Initial time and final time

Remark 5.1 We state the next properties on the pseudo-system f :

$$\forall u \in \mathfrak{S}^{(m)}; \forall x \in f(u) \setminus S^{(n)}; \exists x^1 \in B^n; \exists t_0 \in \mathbb{R}; \forall t < t_0; x(t) = x^1 \quad (11)$$

$$\exists u \in \mathfrak{S}^{(m)}; \exists t_0 \in \mathbb{R}; \exists x \in f(u) \setminus S^{(n)}; \exists \tau \in \mathbb{B}^n; \exists t < t_0; x(t) = \tau \quad (12)$$

$$\exists t_0 \in \mathbb{R}; \exists u \in \mathfrak{S}^{(m)}; \exists x \in f(u) \setminus S^{(n)}; \exists \tau \in \mathbb{B}^n; \exists t < t_0; x(t) = \tau \quad (13)$$

$$\exists u \in \mathfrak{S}^{(m)}; \exists x \in f(u) \setminus S^{(n)}; \exists \tau \in \mathbb{B}^n; \exists t_f \in \mathbb{R}; \exists t > t_f; x(t) = \tau \quad (14)$$

$$\exists u \in \mathfrak{S}^{(m)}; \exists t_f \in \mathbb{R}; \exists x \in f(u) \setminus S^{(n)}; \exists \tau \in \mathbb{B}^n; \exists t > t_f; x(t) = \tau \quad (15)$$

$$\exists t_f \in \mathbb{R}; \exists u \in \mathfrak{S}^{(m)}; \exists x \in f(u) \setminus S^{(n)}; \exists \tau \in \mathbb{B}^n; \exists t > t_f; x(t) = \tau \quad (16)$$

where in (11) τ and t_0 depend on x only, making $\exists \tau \in \mathbb{B}^n; \exists t_0 \in \mathbb{R}$ commute and the situation is similar for τ and t_f in (14).

The properties (11) and (14) are fulfilled by all the pseudo-systems and they are present here for the symmetry of the exposure only.

The dualities between (11) and (14); (12) and (15); (13) and (16) take place and the next implications hold:

$$(13) \Rightarrow (12) \Rightarrow (11)$$

$$(16) \Rightarrow (15) \Rightarrow (14)$$

If f is the null pseudo-system or, more generally, if in one of (11),..., (13) $\exists u \in \mathfrak{S}^{(m)}; f(u) \setminus S^{(n)} = \emptyset$, that property is trivially fulfilled. Here the similarity with Remark 4.1 ends, since defining a partial function $\mathfrak{S}^{(n)} \rightarrow \mathbb{R}$ for example in the case of (11) associating to each state $x \in f(u) \setminus S^{(n)}$ its initial time is not quite natural. Reasoning is the same for the final time.

Definition 5.2 If f satisfies (11), we say that it has unbounded initial time and any t_0 satisfying this property is called unbounded initial time (instant).

Definition 5.3 Let f fulfilling the property (12). We say that it has bounded initial time and any t_0 making this property true is called bounded initial time (instant).

Definition 5.4 When f satisfies (13), we use to say that it has τ -x, or universal initial time and any t_0 fulfilling (13) is called τ -x (or universal) initial time (instant).

Definition 5.5 We suppose that f satisfies (14). Then we say that it has unbounded final time and any t_f satisfying this property is called unbounded final time (instant).

Definition 5.6 If f fulfills the property (15), we say that it has bounded final time. Any number t_f satisfying (15) is called bounded final time (instant).

Definition 5.7 We suppose that the pseudo-system f satisfies the property (16). Then we say that it has τ -x, or universal final time and any number t_f satisfying (16) is called τ -x, or universal final time (instant).

Theorem 5.8 If the pseudo-system f has initial states, then the next non-exclusive possibilities exist:

a) f has initial states and unbounded initial time

$$u \in \mathcal{S}^{(m)}; x \in f(u); \exists t_0 \in \mathbb{R}; \forall t < t_0; x(t) = \bar{x}$$

where \bar{x} and t_0 depend on x only, thus $\exists t_0 \in \mathbb{R}; \forall t < t_0$ commute

b) f has initial states and bounded initial time

$$u \in \mathcal{S}^{(m)}; \exists t_0 \in \mathbb{R}; x \in f(u); \exists t_0 \in \mathbb{R}; \forall t < t_0; x(t) = \bar{x}$$

c) f has initial states and \bar{x} initial time

$$\exists t_0 \in \mathbb{R}; u \in \mathcal{S}^{(m)}; x \in f(u); \exists t_0 \in \mathbb{R}; \forall t < t_0; x(t) = \bar{x}$$

d) f has race-free initial states and unbounded initial time

$$u \in \mathcal{S}^{(m)}; \exists t_0 \in \mathbb{R}; x \in f(u); \exists t_0 \in \mathbb{R}; \forall t < t_0; x(t) = \bar{x}$$

e) f has race-free initial states and bounded initial time

$$u \in \mathcal{S}^{(m)}; \exists t_0 \in \mathbb{R}; \exists t_0 \in \mathbb{R}; x \in f(u); \forall t < t_0; x(t) = \bar{x}$$

where \bar{x} and t_0 depend on u only, thus $\exists t_0 \in \mathbb{R}; \forall t < t_0$ commute

f) f has race-free initial states and \bar{x} initial time

$$\exists t_0 \in \mathbb{R}; u \in \mathcal{S}^{(m)}; \exists t_0 \in \mathbb{R}; x \in f(u); \forall t < t_0; x(t) = \bar{x}$$

g) f has a constant initial state and unbounded initial time

$$\exists t_0 \in \mathbb{R}; u \in \mathcal{S}^{(m)}; x \in f(u); \exists t_0 \in \mathbb{R}; \forall t < t_0; x(t) = \bar{x}$$

h) f has a constant initial state and bounded initial time

$$\exists t_0 \in \mathbb{R}; u \in \mathcal{S}^{(m)}; \exists t_0 \in \mathbb{R}; x \in f(u); \forall t < t_0; x(t) = \bar{x}$$

i) f has a constant initial state and \bar{x} initial time

$$\exists t_0 \in \mathbb{R}; \exists t_0 \in \mathbb{R}; u \in \mathcal{S}^{(m)}; x \in f(u); \forall t < t_0; x(t) = \bar{x}$$

where $\exists t_0 \in \mathbb{R}; \forall t < t_0$ commute.

Proof. e) We must show that the conjunction of (6) and (12) on one hand and

$$u \in \mathcal{S}^{(m)}; \exists t_0 \in \mathbb{R}; \exists t_0 \in \mathbb{R}; x \in f(u); \forall t < t_0; x(t) = \bar{x} \quad (17)$$

where \bar{x} and t_0 depend on u only (making $\exists t_0 \in \mathbb{R}; \forall t < t_0$ commute) on the other hand - are equivalent. This fact is obvious if f is null, thus we can suppose that f is non null and it is sufficient to consider some admissible arbitrary $\bar{x} \in \mathcal{S}^{(m)}$.

(6) and (12) \Rightarrow (17)

From (6) we have the existence of a unique $\varphi^1 \in B^n$ depending on u so that $\exists x \in f(u); x(j-1) + 0 = \varphi^1$ from where $f(u) \cap S^{(n)}$ and $f(u) \setminus S^{(n)} = f(u)$. From (12) we infer that

$$\exists t_0 \in \mathbb{R}; \exists x \in f(u); \forall t < t_0; x(t) = \varphi^1$$

where t_0 depends on u and the statements

$$\exists t_0 \in \mathbb{R}; \exists \varphi^1 \in B^n; \exists x \in f(u); \forall t < t_0; x(t) = \varphi^1$$

$$\exists \varphi^1 \in B^n; \exists t_0 \in \mathbb{R}; \exists x \in f(u); \forall t < t_0; x(t) = \varphi^1$$

are both true, as φ^1 and t_0 depend on u only. (17) is true.

(17) \Rightarrow (6) and (12)

(17) \Rightarrow (6) is obvious. On the other hand a unique $\varphi^1 \in B^n$ exists so that

$$\exists t_0 \in \mathbb{R}; \exists x \in f(u); \forall t < t_0; x(t) = \varphi^1$$

in particular the statement

$$\exists t_0 \in \mathbb{R}; \exists x \in f(u) \setminus S^{(n)}; \forall t < t_0; x(t) = \varphi^1$$

is true, as well as

$$\exists t_0 \in \mathbb{R}; \exists x \in f(u) \setminus S^{(n)}; \exists \varphi^1 \in B^n; \forall t < t_0; x(t) = \varphi^1$$

i.e. (12). ■

Theorem 5.9 The next non-exclusive possibilities exist for the absolutely stable pseudo-system f :

a) f is absolutely stable with unbounded \bar{t} -nal time:

$$\exists u \in \mathfrak{S}^{(m)}; \exists x \in f(u); \exists \varphi^1 \in B^n; \exists t_f \in \mathbb{R}; \forall t > t_f; x(t) = \varphi^1$$

where φ^1 and t_f depend on x only, thus $\exists \varphi^1 \in B^n; \exists t_f \in \mathbb{R}$ commute

b) f is absolutely stable with bounded \bar{t} -nal time:

$$\exists u \in \mathfrak{S}^{(m)}; \exists t_f \in \mathbb{R}; \exists x \in f(u); \exists \varphi^1 \in B^n; \forall t > t_f; x(t) = \varphi^1$$

c) f is absolutely stable with \bar{x} -nal time:

$$\exists t_f \in \mathbb{R}; \exists u \in \mathfrak{S}^{(m)}; \exists x \in f(u); \exists \varphi^1 \in B^n; \forall t > t_f; x(t) = \varphi^1$$

d) f is absolutely race-free stable with unbounded \bar{t} -nal time:

$$\exists u \in \mathfrak{S}^{(m)}; \exists \varphi^1 \in B^n; \exists x \in f(u); \exists t_f \in \mathbb{R}; \forall t > t_f; x(t) = \varphi^1$$

e) f is absolutely race-free stable with bounded \bar{t} -nal time:

$$\exists u \in \mathfrak{S}^{(m)}; \exists \varphi^1 \in B^n; \exists t_f \in \mathbb{R}; \exists x \in f(u); \forall t > t_f; x(t) = \varphi^1$$

where τ and t_f depend on u only, thus $\tau \in \mathbb{B}^n; t_f \in \mathbb{R}$ commute

f) f is absolutely race-free stable with τ -nal time:

$$\tau \in \mathbb{R}; u \in \mathfrak{S}^{(m)}; \tau \in \mathbb{B}^n; x \in f(u); \tau > t_f; x(\tau) = \tau$$

g) f is absolutely constantly stable with unbounded τ -nal time:

$$\tau \in \mathbb{B}^n; u \in \mathfrak{S}^{(m)}; x \in f(u); \tau \in \mathbb{R}; \tau > t_f; x(\tau) = \tau$$

h) f is absolutely constantly stable with bounded τ -nal time:

$$\tau \in \mathbb{B}^n; u \in \mathfrak{S}^{(m)}; \tau \in \mathbb{R}; x \in f(u); \tau > t_f; x(\tau) = \tau$$

i) f is absolutely constantly stable with τ -nal time:

$$\tau \in \mathbb{B}^n; \tau \in \mathbb{R}; u \in \mathfrak{S}^{(m)}; x \in f(u); \tau > t_f; x(\tau) = \tau$$

where τ and t_f are independent on each other, thus $\tau \in \mathbb{B}^n; t_f \in \mathbb{R}$ commute.

Remark 5.10 All the pseudo-systems have unbounded initial (τ -nal) time, the problem is if they have initial (τ -nal) states or not. On the other hand, at both previous theorems, the next implications hold:

$$\begin{array}{ccc} \text{i)} \Rightarrow & \text{h)} \Rightarrow & \text{g)} \\ + & + & + \\ \text{f)} \Rightarrow & \text{e)} \Rightarrow & \text{d)} \\ + & + & + \\ \text{c)} \Rightarrow & \text{b)} \Rightarrow & \text{a)} \end{array}$$

6 Initial state function and τ -nal state function

Definition 6.1 Let the pseudo-system $f : \mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathfrak{S}^{(n)})$. If it has initial states, the function $\hat{A}_0 : \mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathbb{B}^n)$ that is defined by

$$u \in \mathfrak{S}^{(m)}; \hat{A}_0(u) = \{x \in \mathbb{B}^n \mid x \in f(u)\}$$

is called the initial state function of f and the set

$$E_0 = \bigcup_{u \in \mathfrak{S}^{(m)}} \hat{A}_0(u)$$

is called the set of the initial states of f .

Definition 6.2 Let the pseudo-system f . If it has τ -nal states, the function $\hat{A}_f : \mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathbb{B}^n)$ that is given by

$$u \in \mathfrak{S}^{(m)}; \hat{A}_f(u) = \{x \in \mathbb{B}^n \mid x \in f(u)\}$$

is called the τ -nal state function of f and the set

$$E_f = \bigcup_{u \in \mathfrak{S}^{(m)}} \hat{A}_f(u)$$

is called the set of the τ -nal states of f .

Example 6.3 The constant function $\mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathfrak{S}^{(n)})$ equal with f^1g is a pseudo-system with a constant initial state 1 and \bar{x} initial time and it is also absolutely constantly stable with \bar{x} final time. $\hat{A}_0; \mathcal{E}_0; \hat{A}_f; \mathcal{E}_f$ are all defined and equal with f^1g .

Theorem 6.4 Let f a pseudo-system with initial states.

a) If its initial states are race-free, then $\exists u \in \mathfrak{S}^{(m)}; \hat{A}_0(u)$ has at most one element.

b) If f has a constant initial state 1 , then $\hat{A}_0(u) = f^1g$ is true for any admissible u ; for $f = \bar{}$; we have $\mathcal{E}_0 = \bar{}$; and for $f \in \bar{}$; we have $\mathcal{E}_0 = f^1g$.

Proof. a) We suppose that f has race-free initial states and let $u \in \mathfrak{S}^{(m)}$: If $f(u) = \bar{}$, then $\hat{A}_0(u) = \bar{}$; and if $f(u) \in \bar{}$, then a unique $^1 \in \mathcal{B}^n$ exists, depending on u so that $\hat{A}_0(u) = f^1g$.

b) We suppose that f has a constant initial state 1 . If f is null then $\exists u \in \mathfrak{S}^{(m)}; \hat{A}_0(u) = \bar{}$; and $\mathcal{E}_0 = \bar{}$; otherwise for all admissible u we have $\hat{A}_0(u) = f^1g$, the constant function thus $\mathcal{E}_0 = f^1g$. ■

Theorem 6.5 We consider the pseudo-system f with final states.

a) If its final states are race-free, then $\exists u \in \mathfrak{S}^{(m)}; \hat{A}_f(u)$ has at most one element.

b) If f has a constant final state 1 , then $\hat{A}_f(u) = f^1g$ is true for any admissible u ; if admissible inputs do not exist then $\mathcal{E}_f = \bar{}$; and if admissible inputs exist then $\mathcal{E}_f = f^1g$.

7 Pseudo-subsystems

Definition 7.1 The pseudo-systems $f; g : \mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathfrak{S}^{(n)})$ are given. If

$$\exists u \in \mathfrak{S}^{(m)}; f(u) \subseteq g(u)$$

then f is called a pseudo-subsystem of g and the usual notation is $f \subseteq g$.

Remark 7.2 Intuitively, the fact that f is a pseudo-subsystem of g shows that the modeling of a circuit is made more precisely by f than by g , by considering a smaller set of admissible inputs, perhaps. \subseteq is a relation of partial order between $\mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathfrak{S}^{(n)})$ pseudo-systems, where the first element is the null pseudo-system and the last element is the total pseudo-system, see Remark 3.2.

Theorem 7.3 Let the pseudo-system g and $f \subseteq g$ an arbitrary pseudo-subsystem. If g has initial states (race-free initial states, constant initial state), then f has initial states (race-free initial states, constant initial state).

Proof. If one of the previous properties is true for the states in $g(u)$, then it is true for the states in the subset $f(u) \subseteq g(u)$ also, $u \in \mathfrak{S}^{(m)}$. ■

Theorem 7.4 Let $f \leq g$. If g has \bar{t} -nal states (race-free \bar{t} -nal states, constant \bar{t} -nal state), then f has \bar{t} -nal states (race-free \bar{t} -nal states, constant \bar{t} -nal state).

Theorem 7.5 The pseudo-systems $f \leq g$ are given. If g has unbounded initial time (bounded initial time, universal initial time), then f has unbounded initial time (bounded initial time, universal initial time).

Proof. Like previously, if one of the above properties is true for the states in $g(u)$; then it is true for the states in $f(u) \leq g(u)$; $u \in \mathfrak{S}^{(m)}$. ■

Theorem 7.6 Let f be a pseudo-subsystem of g : If g has unbounded \bar{t} -nal time (bounded \bar{t} -nal time, universal \bar{t} -nal time), then f has unbounded \bar{t} -nal time (bounded \bar{t} -nal time, universal \bar{t} -nal time).

Theorem 7.7 If g has initial states and $f \leq g$, then we note with $\circ_0 : \mathfrak{S}^{(m)} \rightarrow P(B^n)$ the initial state function of g and with $i_0 \subseteq B^n$ the set of the initial states of g . We have $\exists u \in \mathfrak{S}^{(m)}; \hat{A}_0(u) \leq \circ_0(u)$ and $\exists_0 \subseteq i_0$.

Proof. f has initial states from Theorem 7.3, thus \exists_0 and \exists_0 exist. Moreover, as $\exists u \in \mathfrak{S}^{(m)}; f(u) \leq g(u)$, the initial values of the states in $f(u)$ are contained between the initial values of the states in $g(u)$, $\hat{A}_0(u) \leq \circ_0(u)$ making $\exists_0 \subseteq i_0$ true too. ■

Theorem 7.8 If g has \bar{t} -nal states and $f \leq g$, we note with $\circ_f : \mathfrak{S}^{(m)} \rightarrow P(B^n)$ the \bar{t} -nal state function of g and with $i_f \subseteq B^n$ the set of the \bar{t} -nal states of g . We have $\exists u \in \mathfrak{S}^{(m)}; \hat{A}_f(u) \leq \circ_f(u)$ and $\exists_f \subseteq i_f$.

8 Dual pseudo-systems

Notation 8.1 For any $s \in B^m; u \in \mathfrak{S}^{(m)}$ we note with $\bar{s} \in B^m; \bar{u} \in \mathfrak{S}^{(m)}$ the complements of $s; u$ made coordinatewise:

$$\bar{s} = (\bar{s}_1; \dots; \bar{s}_m)$$

$$\bar{u}(t) = (\bar{u}_1(t); \dots; \bar{u}_m(t))$$

Definition 8.2 Let the pseudo-system $f : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$. The pseudo-system $f^{\#} : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$ that is defined by

$$\exists u \in \mathfrak{S}^{(m)}; f^{\#}(u) = \{x \mid \exists x \in f(\bar{u})g\}$$

is called the dual pseudo-system of f .

Remark 8.3 We add to the types of duality that were previously presented the duality between $0; 1 \in B$ that gives Definition 8.2. The dual pseudo-system $f^{\#}$ has many properties that can be inferred from those of f .

Theorem 8.4 $(f^{\pi})^{\pi} = f$.

Theorem 8.5 The next statements are equivalent for the pseudo-system f :

- a) f has initial states (race-free initial states, constant initial state)
- b) f^{π} has initial states (race-free initial states, constant initial state).

Proof. We show that f has race-free initial states () f^{π} has race-free initial states:

$$\forall u \in \mathfrak{S}^{(m)}; \exists t_0 \in \mathbb{R}; \forall t < t_0; x(t) = 1 \quad ()$$

$$\forall u \in \mathfrak{S}^{(m)}; \exists t_0 \in \mathbb{R}; \forall t < t_0; x(t) = 1 \quad ()$$

$$\forall u \in \mathfrak{S}^{(m)}; \exists t_0 \in \mathbb{R}; \forall t < t_0; x(t) = 1 \quad ()$$

$$\forall u \in \mathfrak{S}^{(m)}; \exists t_0 \in \mathbb{R}; \forall t < t_0; x(t) = 1$$

■

Theorem 8.6 For the pseudo-system f , the next statements are equivalent:

- a) f has \bar{c} -nal states (race-free \bar{c} -nal states, constant \bar{c} -nal state)
- b) f^{π} has \bar{c} -nal states (race-free \bar{c} -nal states, constant \bar{c} -nal state).

Theorem 8.7 The next properties are equivalent for f :

- a) f has unbounded initial time (bounded initial time, \bar{c} -x initial time)
- b) f^{π} has unbounded initial time (bounded initial time, \bar{c} -x initial time).

Theorem 8.8 Let the pseudo-system f . The next properties are equivalent:

- a) f has unbounded \bar{c} -nal time (bounded \bar{c} -nal time, \bar{c} -x \bar{c} -nal time)
- b) f^{π} has unbounded \bar{c} -nal time (bounded \bar{c} -nal time, \bar{c} -x \bar{c} -nal time).

Theorem 8.9 If f has initial states, we note with $\hat{A}_0^{\pi} : \mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathbb{B}^n)$ the initial state function of f^{π} and with E_0^{π} the set of the initial states of f^{π} . We have

$$\forall u \in \mathfrak{S}^{(m)}; \hat{A}_0^{\pi}(u) = f^{\pi j^{-1}} \hat{A}_0(u)g$$

$$E_0^{\pi} = f^{\pi j^{-1}} E_0 g$$

Proof. If f has initial states, then f^{π} has initial states (Theorem 8.5) thus \hat{A}_0^{π} and E_0^{π} exist. The statements of the theorem are obtained from the fact that

$$\begin{aligned} \forall u \in \mathfrak{S}^{(m)}; \hat{A}_0^{\pi}(u) &= f^{\pi(j^{-1} + 0)} x \hat{A}_0(u)g = \\ &= \overline{f^{\pi(j^{-1} + 0)}} x \hat{A}_0(u)g = \overline{f^{\pi(j^{-1} + 0)}} x \hat{A}_0(u)g = f^{\pi j^{-1}} \hat{A}_0(u)g \end{aligned}$$

■

Theorem 8.10 If f has \bar{c} -nal states, we note with $\hat{A}_f^{\pi} : \mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathbb{B}^n)$ the \bar{c} -nal state function of f^{π} and with E_f^{π} the set of the \bar{c} -nal states of f^{π} . We have

$$\forall u \in \mathfrak{S}^{(m)}; \hat{A}_f^{\pi}(u) = f^{\pi j^{-1}} \hat{A}_f(u)g$$

$$E_f^{\pi} = f^{\pi j^{-1}} E_f g$$

Theorem 8.11 For the pseudo-systems $f, g : \mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathfrak{S}^{(n)})$ we have $f \leq g \iff f^* \leq g^*$

Proof. We get the next sequence of equivalencies:

$$\begin{aligned} & \exists u \in \mathfrak{S}^{(m)}; f(u) \leq g(u) \iff \exists u \in \mathfrak{S}^{(m)}; f(x) \leq f(u) \leq g(u) \leq g(x) \\ \iff & \exists u \in \mathfrak{S}^{(m)}; f(x) \leq f(u) \leq g(u) \leq g(x) \iff \exists u \in \mathfrak{S}^{(m)}; f^*(x) \leq g^*(x) \\ \iff & \exists u \in \mathfrak{S}^{(m)}; f^*(u) \leq g^*(u) \iff \exists u \in \mathfrak{S}^{(m)}; f^*(u) \leq g^*(u) \end{aligned}$$

■

9 Inverse pseudo-systems

Definition 9.1 Let $f : \mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathfrak{S}^{(n)})$. The pseudo-system $f^{-1} : \mathfrak{S}^{(n)} \rightarrow \mathcal{P}(\mathfrak{S}^{(m)})$, called the inverse of f , is defined by

$$\exists x \in \mathfrak{S}^{(n)}; f^{-1}(x) = \{u \in \mathfrak{S}^{(m)}; x \leq f(u)\}$$

Remark 9.2 The idea of construction of f^{-1} is that of inverting the cause-effect relation: it associates to each possible effect x these admissible inputs u that could have caused it. We observe that $u \in f^{-1}(x) \iff x \leq f(u)$.

Example 9.3 The inverse of the null pseudo-system f is the null pseudo-system and the inverse of the total pseudo-system is the total pseudo-system.

Theorem 9.4 For the pseudo-system f we have $(f^{-1})^{-1} = f$.

Proof. For any $u \in \mathfrak{S}^{(m)}$, we can write that

$$(f^{-1})^{-1}(u) = \{x \in \mathfrak{S}^{(n)}; u \in f^{-1}(x)\} = \{x \in \mathfrak{S}^{(n)}; u \leq f(x)\} = f(u)$$

■

Theorem 9.5 If f^{-1} has initial states, then the admissible inputs of f are signals.

Proof. We suppose the contrary, i.e. some admissible input u^0 of f exists that is not a signal:

$$u^0 \in \mathfrak{S}^{(m)} \text{ and } f(u^0) \neq \emptyset; \text{ and } \exists t_0 \in \mathbb{R}; \forall t < t_0; u(t) = \emptyset$$

We take some $x^0 \in f(u^0)$, meaning that $u^0 \in f^{-1}(x^0)$. In the statement relative to the initial states of f^{-1} :

$$\exists x \in \mathfrak{S}^{(n)}; \exists u \in f^{-1}(x); \exists t_0 \in \mathbb{R}; \forall t < t_0; u(t) = \emptyset$$

we have for $x = x^0$ and $u = u^0$:

$$x^0 \in \mathfrak{S}^{(n)} \implies (u^0 \in f^{-1}(x^0) \implies \exists t_0 \in \mathbb{R}; \forall t < t_0; u(t) = \emptyset)$$

The two prerequisites are true and the conclusion is false, contradiction. ■

Theorem 9.6 We suppose that f^{i-1} has initial states and we note with $\hat{A}_0^{i-1} : \mathfrak{S}^{(n)} \rightarrow P(\mathfrak{B}^m)$; \hat{E}_0^{i-1} its initial state function and respectively its set of initial states. We have

$$\begin{aligned} \delta x \in \mathfrak{S}^{(n)}; \hat{A}_0^{i-1}(x) = f u_{(i-1+0)} j u \in \mathfrak{S}^{(m)}; x \in f(u)g \\ \hat{E}_0^{i-1} = f u_{(i-1+0)} j u \in \mathfrak{S}^{(m)}; f(u) \in g \end{aligned}$$

Theorem 9.7 We suppose that f^{i-1} has final states and we note with $\hat{A}_f^{i-1} : \mathfrak{S}^{(n)} \rightarrow P(\mathfrak{B}^m)$; \hat{E}_f^{i-1} its final state function and respectively its set of final states. We have

$$\begin{aligned} \delta x \in \mathfrak{S}^{(n)}; \hat{A}_f^{i-1}(x) = f u_{(1-i-0)} j u \in \mathfrak{S}^{(m)}; x \in f(u)g \\ \hat{E}_f^{i-1} = f u_{(1-i-0)} j u \in \mathfrak{S}^{(m)}; f(u) \in g \end{aligned}$$

Theorem 9.8 If $f \frac{1}{2} g$, then $f^{i-1} \frac{1}{2} g^{i-1}$ and $(f^a)^{i-1} \frac{1}{2} (g^a)^{i-1}$ take place.

Proof. $\delta u \in \mathfrak{S}^{(m)}; f(u) \frac{1}{2} g(u)$ implies

$$\begin{aligned} \delta u \in \mathfrak{S}^{(m)}; \delta x \in \mathfrak{S}^{(n)}; x \in f(u) \Rightarrow x \in g(u) \\ \delta u \in \mathfrak{S}^{(m)}; \delta x \in \mathfrak{S}^{(n)}; u \in f^{i-1}(x) \Rightarrow u \in g^{i-1}(x) \\ \delta x \in \mathfrak{S}^{(n)}; \delta u \in \mathfrak{S}^{(m)}; u \in f^{i-1}(x) \Rightarrow u \in g^{i-1}(x) \\ \delta x \in \mathfrak{S}^{(n)}; f^{i-1}(x) \frac{1}{2} g^{i-1}(x) \end{aligned}$$

On the other hand $f \frac{1}{2} g$ implies $f^a \frac{1}{2} g^a$ (see Theorem 8.11) and from the previous item we get $(f^a)^{i-1} \frac{1}{2} (g^a)^{i-1}$. ■

Theorem 9.9 $(f^{i-1})^a = (f^a)^{i-1}$.

Proof. We get for all $x \in \mathfrak{S}^{(n)}$ that

$$\begin{aligned} (f^{i-1})^a(x) = f \cup j u \in f^{i-1}(x)g = f \cup j x \in f(u)g = f \cup j x \in f^a(u)g = \\ = f \cup j x \in f^a(u)g = f \cup j u \in (f^a)^{i-1}(x)g = (f^a)^{i-1}(x) \end{aligned}$$

■

10 Direct product

Definition 10.1 We consider the pseudo-systems $f : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$; $f^0 : \mathfrak{S}^{(m^0)} \rightarrow P(\mathfrak{S}^{(n^0)})$. The direct product of f and f^0 is by definition the pseudo-system $f \in f^0 : \mathfrak{S}^{(m+m^0)} \rightarrow P(\mathfrak{S}^{(n+n^0)})$ that is defined in the next manner:

$$\delta(u; u^0) \in \mathfrak{S}^{(m+m^0)}; (f \in f^0)(u; u^0) = f(x; x^0) j (x; x^0) \in \mathfrak{S}^{(n+n^0)}; x \in f(u); x^0 \in f^0(u^0)g$$

where u is the projection of the variable from $\mathfrak{S}^{(m+m^0)}$ on the first m coordinates and u^0 is the projection of the variable from $\mathfrak{S}^{(m+m^0)}$ on the last m^0 coordinates. Similarly, x is the projection of the variable from $\mathfrak{S}^{(n+n^0)}$ on the first n coordinates and x^0 is the projection of the same variable on the last n^0 coordinates.

Remark 10.2 $f \in f^0$ is the pseudo-system representing f and f^0 acting independently on each other. Some sort of problem arises here, from the fact that 'independently on each other' refers to the function $f \in f^0 : \mathfrak{S}^{(m)} \in \mathfrak{S}^{(m^0)} \rightarrow P(\mathfrak{S}^{(n)}) \in P(\mathfrak{S}^{(n^0)})$ and we were forced to make the identifications between $\mathfrak{S}^{(m)} \in \mathfrak{S}^{(m^0)}$ and $\mathfrak{S}^{(m+m^0)}$ and respectively between $P(\mathfrak{S}^{(n)}) \in P(\mathfrak{S}^{(n^0)})$ and $P(\mathfrak{S}^{(n+n^0)})$, in order that $f \in f^0$ be a pseudo-system. This means exactly one time axis (like in $\mathfrak{S}^{(m+m^0)}$ and $P(\mathfrak{S}^{(n+n^0)})$) instead of two (like in $\mathfrak{S}^{(m)} \in \mathfrak{S}^{(m^0)}$ and $P(\mathfrak{S}^{(n)}) \in P(\mathfrak{S}^{(n^0)})$). But in this moment f and f^0 do not quite act 'independently on each other'. Things look like claiming 'time is universal, the same for everybody'.

On the other hand, we can write

$$\delta(u; u^0) \in \mathfrak{S}^{(m+m^0)}; (f \in f^0)(u; u^0) = f(u) \in f^0(u^0)$$

if we accept that the elements of $f(u) \in f^0(u^0)$ belong to $P(\mathfrak{S}^{(n+n^0)})$ (not to $P(\mathfrak{S}^{(n)}) \in P(\mathfrak{S}^{(n^0)})$).

Theorem 10.3 The pseudo-systems f and f^0 have initial states (race-free initial states, constant initial state) if and only if $f \in f^0$ has initial states (race-free initial states, constant initial state).

Proof. For example the conjunction of the statements

$$\exists t_0 \in \mathbb{R}; \exists u \in \mathfrak{S}^{(m)}; \exists x \in f(u); \exists t_0 \in \mathbb{R}; \exists t < t_0; x(t) = 1$$

$$\exists t_0^0 \in \mathbb{R}; \exists u^0 \in \mathfrak{S}^{(m^0)}; \exists x^0 \in f^0(u^0); \exists t_0^0 \in \mathbb{R}; \exists t < t_0^0; x^0(t) = 1^0$$

is equivalent with

$$\exists (1; 1^0) \in \mathbb{B}^{n+n^0}; \exists (u; u^0) \in \mathfrak{S}^{(m+m^0)}; \exists (x; x^0) \in (f \in f^0)(u; u^0);$$

$$\exists t_0'' \in \mathbb{R}; \exists t < t_0''; (x(t); x^0(t)) = (1; 1^0)$$

where we can take $t_0'' = \min(t_0; t_0^0)$ each time. ■

Theorem 10.4 f and f^0 have final states (race-free final states, constant final state) if and only if $f \in f^0$ has final states (race-free final states, constant final state).

Theorem 10.5 Let the pseudo-systems $f; f^0$. The next statements are equivalent:

- a) f and f^0 have unbounded initial time (bounded initial time, ∞ initial time)
- b) $f \in f^0$ has unbounded initial time (bounded initial time, ∞ initial time).

Theorem 10.6 The pseudo-systems f and f^0 have unbounded final time (bounded final time, ∞ final time) if and only if $f \in f^0$ has unbounded final time (bounded final time, ∞ final time).

Theorem 10.7 Let the pseudo-systems f and f^0 defined like before. If they have initial states, we note with $\hat{A}_0; \hat{A}_0^0$ their initial state functions and with $(\hat{A} \in \hat{A}^0)_0 : \mathfrak{S}^{(m+m^0)} \rightarrow P(B^{n+n^0})$ the initial state function of $f \in f^0$. We also note with $\mathcal{E}_0; \mathcal{E}_0^0$ the sets of the initial states of f and f^0 and let $(\mathcal{E} \in \mathcal{E}^0)_0$ the set of the initial states of $f \in f^0$. We have

$$\begin{aligned} \delta(u; u^0) \in \mathfrak{S}^{(m+m^0)}; (\hat{A} \in \hat{A}^0)_0(u; u^0) &= \hat{A}_0(u) \in \hat{A}_0^0(u^0) \\ (\mathcal{E} \in \mathcal{E}^0)_0 &= \mathcal{E}_0 \in \mathcal{E}_0^0 \end{aligned}$$

In the previous equations we have identified $P(B^n) \in P(B^{n^0})$ with $P(B^{n+n^0})$.

Proof. If $f; f^0$ have initial states, then $f \in f^0$ has initial states from Theorem 10.3 thus $(\hat{A} \in \hat{A}^0)_0$ and $(\mathcal{E} \in \mathcal{E}^0)_0$ exist. We obtain

$$\begin{aligned} \delta(u; u^0) \in \mathfrak{S}^{(m+m^0)}; (\hat{A} \in \hat{A}^0)_0(u; u^0) &= f(x_{(j-1)+0}; x^0_{(j-1)+0})j(x; x^0) \in (f \in f^0)(u; u^0)g = \\ &= f(x_{(j-1)+0}; x^0_{(j-1)+0})jx \in f(u); x^0 \in f^0(u^0)g = \\ &= fx_{(j-1)+0}jx \in f(u)g \in fx^0_{(j-1)+0}jx^0 \in f^0(u^0)g = \hat{A}_0(u) \in \hat{A}_0^0(u^0) \\ (\mathcal{E} \in \mathcal{E}^0)_0 &= \left[\begin{array}{c} (\hat{A} \in \hat{A}^0)_0(u; u^0) \\ (u; u^0) \in \mathfrak{S}^{(m+m^0)} \\ u \in \mathfrak{S}^{(m)} \end{array} \right] = \left[\begin{array}{c} \hat{A}_0(u) \in \hat{A}_0^0(u^0) \\ (u; u^0) \in \mathfrak{S}^{(m+m^0)} \\ u^0 \in \mathfrak{S}^{(m^0)} \end{array} \right] = \mathcal{E}_0 \in \mathcal{E}_0^0 \end{aligned}$$

■

Theorem 10.8 If $f; f^0$ have final states, we note with $\hat{A}_f; \hat{A}_f^0$ their final state functions and with $(\hat{A} \in \hat{A}^0)_f : \mathfrak{S}^{(m+m^0)} \rightarrow P(B^{n+n^0})$ the final state function of $f \in f^0$. We also note with $\mathcal{E}_f; \mathcal{E}_f^0$ the sets of the final states of f and f^0 and with $(\mathcal{E} \in \mathcal{E}^0)_f$ the set of the final states of $f \in f^0$. We have

$$\begin{aligned} \delta(u; u^0) \in \mathfrak{S}^{(m+m^0)}; (\hat{A} \in \hat{A}^0)_f(u; u^0) &= \hat{A}_f(u) \in \hat{A}_f^0(u^0) \\ (\mathcal{E} \in \mathcal{E}^0)_f &= \mathcal{E}_f \in \mathcal{E}_f^0 \end{aligned}$$

and the same identification between $P(B^n) \in P(B^{n^0})$ and $P(B^{n+n^0})$ like before has been made.

Theorem 10.9 Let the pseudo-systems $f; g : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$, $f^0; g^0 : \mathfrak{S}^{(m^0)} \rightarrow P(\mathfrak{S}^{(n^0)})$. We have that $f \frac{1}{2} g$ and $f^0 \frac{1}{2} g^0$ if and only if $f \in f^0 \frac{1}{2} g \in g^0$.

Theorem 10.10 For any pseudo-systems $f; f^0$ we have $(f \in f^0)^{\alpha} = f^{\alpha} \in f^{0\alpha}$.

Proof. For any $(u; u^0) \in \mathfrak{S}^{(m+m^0)}$ we can write

$$\begin{aligned} (f \in f^0)^{\alpha}(u; u^0) &= f(\bar{x}; \bar{x}^0)j(x; x^0) \in (f \in f^0)(\bar{u}; \bar{u}^0)g = f(\bar{x}; \bar{x}^0)jx \in f(\bar{u}); x^0 \in f^0(\bar{u}^0)g = \\ &= f(\bar{x}; \bar{x}^0)j\bar{x} \in f^{\alpha}(u); \bar{x}^0 \in f^{0\alpha}(u^0)g = f(x; x^0)jx \in f^{\alpha}(u); x^0 \in f^{0\alpha}(u^0)g = \\ &= f(x; x^0)j(x; x^0) \in (f^{\alpha} \in f^{0\alpha})(u; u^0)g = (f^{\alpha} \in f^{0\alpha})(u; u^0) \end{aligned}$$

■

Theorem 10.11 Let f and f^0 . We have that $(f \in f^0)^{i-1} = f^{i-1} \in f^{0i-1}$.

11 Parallel connection

Definition 11.1 The pseudo-systems $f : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$ and $f^0 : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n^0)})$ are considered. The pseudo-system $(f; f^0) : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n+n^0)})$ that is defined in the next manner

$$\forall u \in \mathfrak{S}^{(m)}; (f; f^0)(u) = \{f(x; x^0) \mid (x; x^0) \in \mathfrak{S}^{(n+n^0)}; x \in f(u); x^0 \in f^0(u)\}$$

is called the parallel connection of the systems f and f^0 .

Remark 11.2 The study of the parallel connection of the pseudo-systems is made in quite similar terms with the study of the direct product of pseudo-systems from the previous section.

The relation between the direct product and the parallel connection is expressed by the commutativity of the next diagram

$$\begin{array}{ccc} \mathfrak{S}^{(m)} & \xrightarrow{(f; f^0)} & P(\mathfrak{S}^{(n+n^0)}) \\ \downarrow \Phi \# & \text{iiiiiii!} & \downarrow \text{jj} \\ \mathfrak{S}^{(2m)} & \xrightarrow{\text{ii f i i f i}} & P(\mathfrak{S}^{(n+n^0)}) \end{array}$$

where we have noted with Φ the diagonal function

$$\forall u \in \mathfrak{S}^{(m)}; \Phi(u) = (u; u)$$

12 Serial connection

Definition 12.1 Let the pseudo-systems $f : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$ and $h : \mathfrak{S}^{(n)} \rightarrow P(\mathfrak{S}^{(p)})$. The pseudo-system $h \pm f : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(p)})$ that is defined in the next way

$$\forall u \in \mathfrak{S}^{(m)}; (h \pm f)(u) = \{f(y) \mid y \in f(u); y \in h(x)\}$$

is called the serial connection of the pseudo-systems h and f .

Theorem 12.2 Let the pseudo-systems $f; h$. If h has initial states (constant initial state), then $h \pm f$ has initial states (constant initial state).

Proof. For example from

$$\forall u \in \mathfrak{S}^{(m)}; \forall y \in (h \pm f)(u); \exists x \in f(u); y \in h(x)$$

$$\exists t_0 \in \mathbb{R}^p; \forall x \in \mathfrak{S}^{(n)}; \forall y \in h(x); \exists t_0 \in \mathbb{R}; \forall t < t_0; y(t) = t_0$$

we infer

$$\exists t_0 \in \mathbb{R}^p; \forall u \in \mathfrak{S}^{(m)}; \forall y \in (h \pm f)(u); \exists t_0 \in \mathbb{R}; \forall t < t_0; y(t) = t_0$$

■

Theorem 12.3 If h has \bar{u} -nal states (constant \bar{u} -nal state), then $h \pm f$ has \bar{u} -nal states (constant \bar{u} -nal state).

Theorem 12.4 Let the systems f and h . If h has unbounded initial time (\bar{u} -x initial time), then $h \pm f$ has unbounded initial time (\bar{u} -x initial time).

Proof. For example from

$$\forall u \in \mathcal{S}^{(m)}; \exists y \in (h \pm f)(u) \setminus S^{(p)}; \exists x \in f(u); y \in h(x) \setminus S^{(p)}$$

$$\exists t_0 \in \mathbb{R}; \exists x \in \mathcal{S}^{(m)}; \exists y \in h(x) \setminus S^{(p)}; \exists \varphi \in \mathbb{B}^p; \forall t < t_0; y(t) = \varphi$$

we get

$$\exists t_0 \in \mathbb{R}; \exists u \in \mathcal{S}^{(m)}; \exists y \in (h \pm f)(u) \setminus S^{(p)}; \exists \varphi \in \mathbb{B}^p; \forall t < t_0; y(t) = \varphi$$

■

Theorem 12.5 If h has unbounded \bar{u} -nal time (\bar{u} -x \bar{u} -nal time), then $h \pm f$ has unbounded \bar{u} -nal time (\bar{u} -x \bar{u} -nal time).

Theorem 12.6 We consider the pseudo-systems f and h . If h has initial states, we note with $\bar{u}_0; \pm_0$ on one hand and Φ_0 on the other hand the initial state functions of $h; h \pm f$, respectively the set of initial states of $h \pm f$. The next formulas are true:

$$\forall u \in \mathcal{S}^{(m)}; \pm_0(u) = \left[\begin{array}{c} \bar{u}_0(x) \\ x \in f(u) \end{array} \right]$$

$$\Phi_0 = \left[\begin{array}{c} \left[\begin{array}{c} \bar{u}_0(x) \\ x \in f(u) \end{array} \right] \\ u \in \mathcal{S}^{(m)} \end{array} \right]$$

Theorem 12.7 Let the pseudo-systems f and h . We suppose that h has \bar{u} -nal states and we use the notations $\bar{u}_f; \pm_f$ on one hand and Φ_f on the other hand for the \bar{u} -nal state functions of $h; h \pm f$, respectively for the set of \bar{u} -nal states of $h \pm f$. The next formulas are true:

$$\forall u \in \mathcal{S}^{(m)}; \pm_f(u) = \left[\begin{array}{c} \bar{u}_f(x) \\ x \in f(u) \end{array} \right]$$

$$\Phi_f = \left[\begin{array}{c} \left[\begin{array}{c} \bar{u}_f(x) \\ x \in f(u) \end{array} \right] \\ u \in \mathcal{S}^{(m)} \end{array} \right]$$

Proof. From the fact that h has \bar{u} -nal states we infer, see Theorem 12.3, that $h \pm f$ has \bar{u} -nal states so that $\pm_f; \Phi_f$ exist. We have:

$$\forall u \in \mathcal{S}^{(m)}; \pm_f(u) = f(y(1; 0)) \cap (h \pm f)(u) = f(y(1; 0)) \cap \{x \in f(u) \mid y \in h(x)\} =$$

$$= \left[\begin{array}{c} f(y(1; 0)) \cap (h \pm f)(u) \\ x \in f(u) \end{array} \right] = \left[\begin{array}{c} \bar{u}_f(x) \\ x \in f(u) \end{array} \right]$$

■

Theorem 12.8 Let's consider the pseudo-systems $f; g : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$ and $h; h_1 : \mathfrak{S}^{(n)} \rightarrow P(\mathfrak{S}^{(p)})$. We have:

- a) $f \sqsubseteq g \Rightarrow h \pm f \sqsubseteq h \pm g$
- b) $h \sqsubseteq h_1 \Rightarrow h \pm f \sqsubseteq h_1 \pm f$

Proof. Let $u \in \mathfrak{S}^{(m)}$ arbitrary. Because $f(u) \sqsubseteq g(u)$ we infer that

$$(h \pm f)(u) = \{y \mid \exists x \in f(u); y \in h(x)\} \sqsubseteq \{y \mid \exists x \in g(u); y \in h(x)\} = (h \pm g)(u)$$

■

Theorem 12.9 For the pseudo-systems f and h , we have $(h \pm f)^{\alpha} = h^{\alpha} \pm f^{\alpha}$.

Proof. For any $u \in \mathfrak{S}^{(m)}$ we have

$$\begin{aligned} (h \pm f)^{\alpha}(u) &= \{y \mid \exists \alpha \in \mathbb{N}; \exists x \in (h \pm f)(u); y \in h^{\alpha}(x)\} \\ &= \{y \mid \exists \alpha \in \mathbb{N}; \exists x \in f(u); y \in h^{\alpha}(x)\} \cup \{y \mid \exists \alpha \in \mathbb{N}; \exists x \in g(u); y \in h^{\alpha}(x)\} \\ &= \{y \mid \exists \alpha \in \mathbb{N}; \exists x \in f^{\alpha}(u); y \in h^{\alpha}(x)\} \cup \{y \mid \exists \alpha \in \mathbb{N}; \exists x \in g^{\alpha}(u); y \in h^{\alpha}(x)\} \\ &= (h^{\alpha} \pm f^{\alpha})(u) \end{aligned}$$

■

Theorem 12.10 For any pseudo-system f we have

$$\begin{aligned} \forall u \in \mathfrak{S}^{(m)}; (f^{i-1} \pm f)(u) &= \{v \mid \exists v \in \mathfrak{S}^{(m)}; f(u) \setminus f(v) \in \mathfrak{E}; \\ & \exists x \in \mathfrak{S}^{(n)}; (f \pm f^{i-1})(x) = f(z) \setminus f^{i-1}(z) \in \mathfrak{E}; g \end{aligned}$$

Proof. We observe that

$$\begin{aligned} \forall u \in \mathfrak{S}^{(m)}; (f^{i-1} \pm f)(u) &= \{v \mid \exists x \in f(u); v \in f^{i-1}(x)\} \cup \\ &= \{v \mid \exists x \in f(u); x \in f(v)\} \cup \{v \mid \exists x \in f(u); x \in f(v)\} \setminus \{v \mid \exists x \in f(u); x \in f(v)\} \in \mathfrak{E}; g \end{aligned}$$

and similarly for the other statement. ■

Theorem 12.11 Let f and h . We have $(h \pm f)^{i-1} = f^{i-1} \pm h^{i-1}$.

Proof. For any $y \in \mathfrak{S}^{(p)}$ we have

$$\begin{aligned} (h \pm f)^{i-1}(y) &= \{x \mid \exists u \in \mathfrak{S}^{(m)}; (h \pm f)(u) \in \mathfrak{E}; \exists x \in \mathfrak{S}^{(n)}; (h \pm f)(u) \in \mathfrak{E}; \\ &= \{x \mid \exists u \in \mathfrak{S}^{(m)}; h(u) \in \mathfrak{E}; \exists x \in \mathfrak{S}^{(n)}; h(u) \in \mathfrak{E}\} \cup \{x \mid \exists u \in \mathfrak{S}^{(m)}; f(u) \in \mathfrak{E}; \exists x \in \mathfrak{S}^{(n)}; f(u) \in \mathfrak{E}\} \\ &= \{x \mid \exists u \in \mathfrak{S}^{(m)}; h(u) \in \mathfrak{E}; \exists x \in \mathfrak{S}^{(n)}; h(u) \in \mathfrak{E}\} \cup \{x \mid \exists u \in \mathfrak{S}^{(m)}; f(u) \in \mathfrak{E}; \exists x \in \mathfrak{S}^{(n)}; f(u) \in \mathfrak{E}\} \\ &= (h^{i-1} \pm f^{i-1})(y) \end{aligned}$$

■

Theorem 12.12 We consider the pseudo-systems $f : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$, $f^0 : \mathfrak{S}^{(m^0)} \rightarrow P(\mathfrak{S}^{(n^0)})$, respectively $h : \mathfrak{S}^{(n)} \rightarrow P(\mathfrak{S}^{(p)})$, $h^0 : \mathfrak{S}^{(n^0)} \rightarrow P(\mathfrak{S}^{(p^0)})$. The next formula is true:

$$(h \in h^0) \pm (f \in f^0) = (h \pm f) \in (h^0 \pm f^0)$$

Theorem 12.13 If in the hypothesis of the previous theorem we have the special case $m = m^0$, then we can write

$$(h \in h^0)_{\pm}(f; f^0) = (h \pm f; h^0 \pm f^0)$$

Proof. For any $u \in \mathfrak{S}^{(m)}$ we have

$$\begin{aligned} ((h \in h^0)_{\pm}(f; f^0))(u) &= f(y; y^0) \text{ if } (x; x^0) \in (f; f^0)(u) \text{ and } (y; y^0) \in (h \in h^0)(x; x^0) \text{ else} \\ &= f(y; y^0) \text{ if } (x; x^0) \in f(u) \text{ and } (y; y^0) \in h(x) \text{ and } (x; x^0) \in f^0(u) \text{ and } (y; y^0) \in h^0(x^0) \text{ else} \\ &= f(y; y^0) \text{ if } (x; x^0) \in (h \pm f)(u) \text{ and } (y; y^0) \in (h^0 \pm f^0)(u) \text{ else} \\ &= (h \pm f; h^0 \pm f^0)(u) \end{aligned}$$

■

13 Complement

Definition 13.1 Let $f : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$. The pseudo-system $Cf : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$ that is defined by

$$Cu \in \mathfrak{S}^{(m)}; Cf(u) = \mathfrak{S}^{(n)} \setminus f(u)$$

is called the complement of f .

Remark 13.2 Intuitively, if $x \in f(u)$ are these states that model a circuit then $x \in Cf(u)$ are the states that do not model that circuit.

Theorem 13.3 $CCf = f$

Theorem 13.4 If $f; g : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$; then $f \leq g$ if and only if $Cg \leq Cf$

Proof. We have

$$\begin{aligned} f \leq g & \iff \forall u \in \mathfrak{S}^{(m)}; f(u) \subseteq g(u) \\ & \iff \forall u \in \mathfrak{S}^{(m)}; \mathfrak{S}^{(n)} \setminus g(u) \subseteq \mathfrak{S}^{(n)} \setminus f(u) \\ & \iff \forall u \in \mathfrak{S}^{(m)}; Cg(u) \subseteq Cf(u) \\ & \iff Cg \leq Cf \end{aligned}$$

■

Theorem 13.5 $(Cf)^{\#} = Cf^{\#}$

Proof. For any $u \in \mathfrak{S}^{(m)}$ we can write

$$\begin{aligned} (Cf)^{\#}(u) &= \{x \mid x \in (Cf)(\bar{u})\} = \{x \mid x \in \mathfrak{S}^{(n)} \setminus f(\bar{u})\} = \{x \mid x \in \mathfrak{S}^{(n)} \setminus f(\bar{u})\} \\ &= \{x \mid x \in \mathfrak{S}^{(n)} \setminus f(\bar{u})\} = \{x \mid x \in \mathfrak{S}^{(n)} \setminus f(\bar{u})\} \\ &= \{x \mid x \in \mathfrak{S}^{(n)} \setminus f^{\#}(u)\} = (Cf^{\#})(u) \end{aligned}$$

■

Theorem 13.6 $(Cf)^i = Cf^i$

Proof. For all $x \in \mathfrak{S}^{(n)}$ we have

$$\begin{aligned} (Cf)^i(x) &= f(u \in \mathfrak{S}^{(m)}; x \in Cf(u))g = f(u \in \mathfrak{S}^{(m)}; x \in \mathfrak{S}^{(n)} \cap f(u))g = \\ &= f(u \in \mathfrak{S}^{(m)} \cap f^{-1}(x))g = f(u \in \mathfrak{S}^{(m)} \cap f^{-1}(x))g = (Cf^i)(x) \end{aligned}$$

■

Theorem 13.7 Let $f : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$; $f^0 : \mathfrak{S}^{(m^0)} \rightarrow P(\mathfrak{S}^{(n^0)})$ two pseudo-systems. We have $Cf \in Cf^0 \iff C(f \in f^0)$.

Proof. $\forall (u; u^0) \in \mathfrak{S}^{(m+m^0)}$;

$$\begin{aligned} (Cf \in Cf^0)(u; u^0) &= f(x; x^0) \cap x \in Cf(u); x^0 \in Cf^0(u^0)g = \\ &= f(x; x^0) \cap x \in \mathfrak{S}^{(n)} \text{ and } x \in f(u) \text{ and } x^0 \in \mathfrak{S}^{(n^0)} \text{ and } x^0 \in f^0(u^0)g = \\ &= f(x; x^0) \cap (x \in \mathfrak{S}^{(n)} \text{ and } x \in f(u) \text{ and } x^0 \in \mathfrak{S}^{(n^0)} \text{ and } x^0 \in f^0(u^0))g = \\ &= f(x; x^0) \cap (x \in \mathfrak{S}^{(n)} \text{ and } (x \in f(u) \text{ or } x^0 \in f^0(u^0)))g = \\ &= f(x; x^0) \cap (x \in \mathfrak{S}^{(n)} \text{ and } (x \in f(u) \text{ or } x^0 \in f^0(u^0)))g = C(f \in f^0)(u; u^0) \end{aligned}$$

■

Theorem 13.8 For $f : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$; $f^0 : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n^0)})$ we can write $(Cf; Cf^0) \iff C(f; f^0)$.

14 Intersection and reunion

Definition 14.1 Let the pseudo-systems $f; g : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$. The pseudo-systems $f \setminus g$; $f \sqcup g : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$ are defined by

$$\forall u \in \mathfrak{S}^{(m)}; (f \setminus g)(u) = f(u) \setminus g(u)$$

$$\forall u \in \mathfrak{S}^{(m)}; (f \sqcup g)(u) = f(u) \sqcup g(u)$$

Remark 14.2 The intersection of the pseudo-systems represents the gain of information (of precision) in the modeling of a circuit by considering the validity of two models at the same time. The reunion of the pseudo-systems is the dual concept representing the loss of information (of precision) in modeling as a result of considering the validity of one of two models.

The set of the $\mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)})$ pseudo-systems is a Boole algebra relative to $C; \setminus; \sqcup$. The zero and the one of this Boole algebra are the null and the total pseudo-systems.

Theorem 14.3 Let the pseudo-systems f and g . If f has initial states (race-free initial states, constant initial state), then $f \setminus g$ has initial states (race-free initial states, constant initial state).

Proof. $f \setminus g \sqsubseteq f$ and the statement of the theorem follows from Theorem 7.3. ■

Theorem 14.4 If f has \bar{c} -nal states (race-free \bar{c} -nal states, constant \bar{c} -nal state), then $f \setminus g$ has \bar{c} -nal states (race-free \bar{c} -nal states, constant \bar{c} -nal state).

Theorem 14.5 If the pseudo-systems $f;g$ have initial states (a common constant initial state), then $f \sqcap g$ has initial states (constant initial state).

Theorem 14.6 If $f;g$ have \bar{c} -nal states (a common constant \bar{c} -nal state), then $f \sqcap g$ has \bar{c} -nal states (constant \bar{c} -nal state).

Remark 14.7 The statements of the previous two theorems are not true in general for the pseudo-systems $f;g$ with race-free initial states and for the pseudo-systems $f;g$ with constant initial states, because it is possible that the two partial functions $\mathcal{S}^{(m)} \rightarrow \mathcal{B}^n$ from Remark 4.1 corresponding to f and g differ, respectively that the two constant initial states corresponding to f and g differ. Similar reasoning for the \bar{c} -nal states. Such 'disappearances of the middle statement', could be the race-free statement about the initial/ \bar{c} -nal states, could be the boundness statement about the initial/ \bar{c} -nal time, have already occurred (for different reasons) at theorems 12.2, ..., 12.5.

Theorem 14.8 If f has unbounded initial time (bounded initial time, \bar{x} initial time), then $f \setminus g$ has unbounded initial time (bounded initial time, \bar{x} initial time).

Proof. $f \setminus g \sqsubseteq f$ and the results follow from Theorem 7.5. ■

Theorem 14.9 If f has unbounded \bar{c} -nal time (bounded \bar{c} -nal time, \bar{x} \bar{c} -nal time), then $f \setminus g$ has unbounded \bar{c} -nal time (bounded \bar{c} -nal time, \bar{x} \bar{c} -nal time).

Theorem 14.10 If $f;g$ have unbounded initial time (bounded initial time, \bar{x} initial time), then $f \sqcap g$ has unbounded initial time (bounded initial time, \bar{x} initial time).

Theorem 14.11 If $f;g$ have unbounded \bar{c} -nal time (bounded \bar{c} -nal time, \bar{x} \bar{c} -nal time), then $f \sqcap g$ has unbounded \bar{c} -nal time (bounded \bar{c} -nal time, \bar{x} \bar{c} -nal time).

Proof. We suppose for example that f and g satisfy (15), i.e. they have bounded \bar{c} -nal time. For some arbitrary $u \in \mathcal{S}^{(m)}$, let $t_f; t_f^0$ the \bar{c} -nal time instants of f , respectively of g . Then (15) is satisfied by $f \sqcap g$ because we can choose for u the \bar{c} -nal time instant $t_f^u \rightarrow \max(t_f; t_f^0)$. ■

Theorem 14.12 We suppose that $f; g$ have initial states. We have $(\hat{A} \setminus \circ)_0; (\hat{A} [\circ])_0 : \mathfrak{S}^{(m)} \rightarrow P(B^n)$;

$$\forall u \in \mathfrak{S}^{(m)}; (\hat{A} \setminus \circ)_0(u) = \hat{A}_0(u) \setminus \circ_0(u)$$

$$\forall u \in \mathfrak{S}^{(m)}; (\hat{A} [\circ])_0(u) = \hat{A}_0(u) [\circ_0(u)$$

$$(\mathcal{E} \setminus i)_0 = \bigcap_{u \in \mathfrak{S}^{(m)}} \hat{A}_0(u) \setminus \circ_0(u)$$

$$(\mathcal{E} [i])_0 = \bigcap_{u \in \mathfrak{S}^{(m)}} \hat{A}_0(u) [\circ_0(u)$$

We have noted with $\hat{A}_0; \circ_0; (\hat{A} \setminus \circ)_0; (\hat{A} [\circ])_0$ the initial state functions of $f; g; f \setminus g; f [g$ and with $(\mathcal{E} \setminus i)_0; (\mathcal{E} [i])_0$ the sets of initial states of $f \setminus g; f [g$.

Proof. $f [g$ has initial states from Theorem 14.5, thus $(\hat{A} [\circ])_0$ and $(\mathcal{E} [i])_0$ exist. We can write that $\forall u \in \mathfrak{S}^{(m)}$;

$$\begin{aligned} (\hat{A} [\circ])_0(u) &= f(x_{i-1} + 0) \times (f [g)(u) = f(x_{i-1} + 0) \times (f(u) [g(u) = \\ &= f(x_{i-1} + 0) \times f(u)g [f(x_{i-1} + 0) \times g(u) = \hat{A}_0(u) [\circ_0(u) \end{aligned}$$

■

Theorem 14.13 If $f; g$ have final states, then we have $(\hat{A} \setminus \circ)_f; (\hat{A} [\circ])_f : \mathfrak{S}^{(m)} \rightarrow P(B^n)$;

$$\forall u \in \mathfrak{S}^{(m)}; (\hat{A} \setminus \circ)_f(u) = \hat{A}_f(u) \setminus \circ_f(u)$$

$$\forall u \in \mathfrak{S}^{(m)}; (\hat{A} [\circ])_f(u) = \hat{A}_f(u) [\circ_f(u)$$

$$(\mathcal{E} \setminus i)_f = \bigcap_{u \in \mathfrak{S}^{(m)}} \hat{A}_f(u) \setminus \circ_f(u)$$

$$(\mathcal{E} [i])_f = \bigcap_{u \in \mathfrak{S}^{(m)}} \hat{A}_f(u) [\circ_f(u)$$

The notations are obvious and similar with those from the previous theorem.

Theorem 14.14 We have

$$(f \setminus g)^n = f^n \setminus g^n$$

$$(f [g)^n = f^n [g^n$$

Proof. $\forall u \in \mathfrak{S}^{(m)}$;

$$\begin{aligned} (f [g)^n(u) &= f \times (f [g)(u) = f \times (f(u) [g(u) = \\ &= f \times f(u)g [f \times f(u)g = f^n(u) [g^n(u) = (f^n [g^n)(u) \end{aligned}$$

■

Theorem 14.15 The next formulas of inversion take place:

$$(f \setminus g)^{i-1} = f^{i-1} \setminus g^{i-1}$$

$$(f [g)^{i-1} = f^{i-1} [g^{i-1}$$

Proof. $\forall x \in \mathfrak{S}^{(n)}$;

$$\begin{aligned} (f \setminus g)^{i-1}(x) &= f \setminus g \setminus \dots \setminus (f \setminus g)(u)g = f \setminus g \setminus \dots \setminus f(u) \setminus g(u)g = \\ &= f \setminus g \setminus \dots \setminus f(u)g \setminus \dots \setminus f \setminus g \setminus \dots \setminus g(u)g = f^{i-1}(x) \setminus g^{i-1}(x) = (f^{i-1} \setminus g^{i-1})(x) \end{aligned}$$

■

Theorem 14.16 Let the pseudo-systems $f; g : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)}); f^0; g^0 : \mathfrak{S}^{(m^0)} \rightarrow P(\mathfrak{S}^{(n^0)})$. The next statements are true:

$$(f \setminus g) \in (f^0 \setminus g^0) = (f \in f^0) \setminus (g \in g^0)$$

$$(f [g) \in (f^0 [g^0) = (f \in f^0) [(g \in g^0)$$

Proof. We have $\forall (u; u^0) \in \mathfrak{S}^{(m+m^0)}$:

$$\begin{aligned} ((f \setminus g) \in (f^0 \setminus g^0))(u; u^0) &= f(x; x^0) \setminus g(u) \text{ and } x^0 \in (f^0 \setminus g^0)(u^0)g = \\ &= f(x; x^0) \setminus g(u) \text{ and } x^0 \in f^0(u^0) \setminus g^0(u^0)g = \\ &= f(x; x^0) \setminus g(u) \text{ and } x^0 \in f^0(u^0) \text{ and } x^0 \in g^0(u^0)g = \\ &= f(x; x^0) \setminus g(u) \text{ and } x^0 \in f^0(u^0) \text{ and } x^0 \in g(u) \text{ and } x^0 \in g^0(u^0)g = \\ &= f(x; x^0) \setminus g(u) \text{ and } x^0 \in f^0(u^0)g \setminus f(x; x^0) \setminus g(u) \text{ and } x^0 \in g^0(u^0)g = \\ &= (f \in f^0)(u; u^0) \setminus (g \in g^0)(u; u^0) = ((f \in f^0) \setminus (g \in g^0))(u; u^0) \end{aligned}$$

■

Theorem 14.17 We consider the pseudo-systems $f; g : \mathfrak{S}^{(m)} \rightarrow P(\mathfrak{S}^{(n)}); f^0; g^0 : \mathfrak{S}^{(m^0)} \rightarrow P(\mathfrak{S}^{(n^0)})$. We have:

$$(f \setminus g; f^0 \setminus g^0) = (f; f^0) \setminus (g; g^0)$$

$$(f [g; f^0 [g^0) = (f; f^0) [(g; g^0)$$

Theorem 14.18 For the pseudo-systems $f; g$ and $h; h_1 : \mathfrak{S}^{(n)} \rightarrow P(\mathfrak{S}^{(p)})$ we have:

$$h \pm (f \setminus g) \setminus (h \pm f) \setminus (h \pm g)$$

$$h \pm (f [g) [(h \pm f) [(h \pm g)$$

$$(h \setminus h_1) \pm f \setminus (h \pm f) \setminus (h_1 \pm f)$$

$$(h [h_1) \pm f [(h \pm f) [(h_1 \pm f)$$

Proof. $\forall u \in \mathfrak{S}^{(m)}$;

$$\begin{aligned} (h \pm (f \setminus g))(u) &= \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid (f \setminus g)(u) \mid y \in h(x) \} = \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid f(u) \setminus g(u) \mid y \in h(x) \} = \\ &= \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid x \in f(u) \text{ and } x \in g(u) \text{ and } y \in h(x) \} = \\ &= \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid x \in f(u) \text{ and } y \in h(x) \text{ and } x \in g(u) \text{ and } y \in h(x) \} = \\ &= \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid x \in f(u) \text{ and } y \in h(x) \} \setminus \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid x \in g(u) \text{ and } y \in h(x) \} = \\ &= (h \pm f)(u) \setminus (h \pm g)(u) = ((h \pm f) \setminus (h \pm g))(u) \end{aligned}$$

$\forall u \in \mathfrak{S}^{(m)}$;

$$\begin{aligned} ((h \circ h_1) \pm f)(u) &= \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid (h \circ h_1)(x) \in y \} = \\ &= \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid x \in f(u) \text{ and } y \in h(x) \circ h_1(x) \} = \\ &= \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid x \in f(u) \text{ and } y \in h_1(x) \} \circ \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid x \in f(u) \text{ and } y \in h(x) \} = \\ &= \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid x \in f(u) \text{ and } y \in h(x) \} \circ \{ \langle x, y \rangle \in \mathfrak{S}^{(m)} \mid x \in f(u) \text{ and } y \in h_1(x) \} = \\ &= (h \pm f)(u) \circ (h_1 \pm f)(u) = ((h \pm f) \circ (h_1 \pm f))(u) \end{aligned}$$

■

15 Systems

Definition 15.1 Let the pseudo-system $f : \mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathfrak{S}^{(n)})$. The set U_f of the admissible inputs defined by

$$U_f = \{ u \in \mathfrak{S}^{(m)} \mid f(u) \neq \emptyset \}$$

is also called the support (set) of f .

Definition 15.2 The (asynchronous) pseudo-system f is called (asynchronous) system if

- $U_f \neq \emptyset$;
- $U_f \subseteq \mathfrak{S}^{(m)}$;
- $\forall u \in U_f; f(u) \subseteq \mathfrak{S}^{(n)}$.

Remark 15.3 We shall identify the system f with the function $f_1 : U \rightarrow \mathcal{P}(\mathfrak{S}^{(n)})$, where $U = U_f$, that is defined by $\forall u \in U; f_1(u) = f(u)$. We shall also identify the initial state function $\hat{A}_0 : \mathfrak{S}^{(m)} \rightarrow \mathcal{P}(\mathfrak{B}^n)$ with the function $\hat{A}_{10} : U \rightarrow \mathcal{P}(\mathfrak{B}^n)$ defined by $\forall u \in U; \hat{A}_{10}(u) = \hat{A}_0(u)$.

Notation 15.4 The systems are noted sometimes with $f : U \rightarrow P^{\alpha}(S^{(n)})$, where $U \subseteq S^{(m)}$ is non-empty. If $\exists u \in U; f(u)$ has a single element, then we have the usual notation $f : U \rightarrow S^{(n)}$ of the uni-valued functions. Similarly, their initial state functions are noted sometimes with $A_0 : U \rightarrow P^{\alpha}(B^n)$ or with $A_0 : U \rightarrow B^n$ when $\exists u \in U; x(j-1+0)$ is unique.

Remark 15.5 The systems are those non-null pseudo-systems f for which the admissible inputs and the possible states are signals (resulting that f has initial states). The concept creates an asymmetry between the initial states and the final states because:

- it is natural that the inputs be considered commands, a deliberate manner of acting on the circuit modeled by f with the purpose of producing a certain effect. But this is made after choosing an initial time instant t_0 from which we order our actions in the increasing sense of the time axis (not in both senses)
- it is natural that we associate to the request $U \subseteq S^{(m)}$ a request (Definition 15.2 c)) that is dual to stability: the system orders its reactions in the increasing sense of the time axis (not in both senses).

Example 15.6 The fact that (4) defines a $S \rightarrow P^{\alpha}(S)$ system is obvious if we observe that

$$\exists u \in S; \exists \zeta \in (0; d]; \int_{\zeta}^{\infty} u(\tau) \cdot u(t - \zeta) \cdot \int_{\zeta}^{\infty} u(\tau) \tau \geq [t_i - d; t]$$

thus $x(t) = u(t - \zeta)$, which is a signal, satisfies it whenever $\zeta \in (0; d]$. This system is called the symmetrical upper bounded, lower unbounded delay.

Notation 15.7 Let $f : S^{(m)} \rightarrow P(S^{(n)})$ a pseudo-system with the property that

$$\exists u \in S^{(m)}; f(u) \setminus S^{(n)} \neq \emptyset; \quad (18)$$

We note with $[f] : U \rightarrow P^{\alpha}(S^{(n)})$ the function that is defined by

$$U = \{u \in S^{(m)}; f(u) \setminus S^{(n)} = \emptyset\}; g \quad (19)$$

$$\exists u \in U; [f](u) = f(u) \setminus S^{(n)} \quad (20)$$

Theorem 15.8 a) $[f]$ is a system

b) $[f] \subseteq f$

c) Let $g : S^{(m)} \rightarrow P(S^{(n)})$ a system so that $g \subseteq f$. Then $g \subseteq [f]$, i.e. $[f]$ is the greatest system that is included in f .

Proof. a) $U \neq \emptyset$; follows from (18) and (19), $U \subseteq S^{(m)}$ is a consequence of (19) and $\exists u \in U; [f](u) \subseteq S^{(n)}$ results from (20), thus $[f]$ is a system.

b) From (20)

c) Let $g : S^{(m)} \rightarrow P(S^{(n)})$ a system so that $\exists u \in S^{(m)}; g(u) \subseteq f(u)$, from where $\exists u \in S^{(m)}; g(u) \setminus S^{(n)} \subseteq f(u) \setminus S^{(n)} = [f](u)$ ■

Definition 15.9 When the pseudo-system f satisfies the property (18), $[f]$ is called the system that is induced by f .

Theorem 15.10 The pseudo-system f is a system if and only if $f = [f]$.

Proof. (\Leftarrow) is obvious, since $[f]$ is a system.

(\Rightarrow) Admissible inputs exist and let u such an input. Because f is a system, u is signal. From $f(u) \in S^{(n)}$, we have that $f(u) = f(u) \setminus S^{(n)}$ and as u was arbitrarily chosen we infer that $f = [f]$. ■

Theorem 15.11 For any system f , the initial state function \hat{A}_0 and the set of the initial states E_0 exist.

Theorem 15.12 Let the systems $f : U \rightarrow P^{\alpha}(S^{(n)}); g : V \rightarrow P^{\alpha}(S^{(n)}); U, V \subseteq P^{\alpha}(S^{(m)})$. We have

$$f \cup g : U \cup V \rightarrow P^{\alpha}(S^{(n)}) \text{ and } \forall u \in U; f(u) \cup g(u)$$

Proof. $f \cup g : U \cup V \rightarrow P^{\alpha}(S^{(n)})$ and $\forall u \in U; f(u) \cup g(u)$

Each of the suppositions $U \cap V \neq \emptyset$; and respectively $\exists u \in U; \exists x \in f(u)$ so that $x \in S^{(n)} \cap g(u)$ gives a contradiction with the hypothesis $f \cup g$

$U \cup V \rightarrow P^{\alpha}(S^{(n)})$ and $\forall u \in U; f(u) \cup g(u) \Rightarrow f \cup g$

The implication is obvious. ■

Theorem 15.13 If f is a system, then its dual f^{α} is a system too.

Theorem 15.14 For the system f , the function $f^{-1} : X \rightarrow P^{\alpha}(S^{(m)})$ given by

$$X = \{x \mid \exists u \in U; x \in f(u)\}$$

$$\forall x \in X; f^{-1}(x) = \{u \in U; x \in f(u)\}$$

is a system that coincides with the inverse of f (as pseudo-system).

Proof. From the hypothesis, the support U of f is non-empty so that we have $X \neq \emptyset$; . The fact that $U \subseteq S^{(m)}$ implies $\forall x \in X; f^{-1}(x) \subseteq S^{(m)}$ and $\forall u \in U; f(u) \subseteq S^{(n)}$ gives $X \subseteq S^{(n)}$, thus f^{-1} is a system. f^{-1} obviously coincides with the inverse of f as pseudo-system. ■

Theorem 15.15 The direct product of two systems is a system.

Proof. We consider the systems $f : U \rightarrow P^{\alpha}(S^{(n)}); U \subseteq P^{\alpha}(S^{(m)})$ and $f^0 : U^0 \rightarrow P^{\alpha}(S^{(n^0)}); U^0 \subseteq P^{\alpha}(S^{(m^0)})$. We remark that $U \in U^0 \subseteq P^{\alpha}(S^{(m+m^0)})$ and $\forall (u; u^0) \in U \times U^0; (f \times f^0)(u; u^0) \in P^{\alpha}(S^{(n+n^0)})$, thus $f \times f^0$ is a system. ■

Theorem 15.16 Let the systems $f : U \rightarrow P^{\alpha}(S^{(n)}); f^0 : U^0 \rightarrow P^{\alpha}(S^{(n^0)}); U; U^0 \subseteq P^{\alpha}(S^{(m)})$. Their parallel connection is a system if and only if $U \cup U^0 \neq \emptyset$; . In this case we have $(f; f^0) : U \cup U^0 \rightarrow P^{\alpha}(S^{(n+n^0)})$;

$$\forall u \in U \cup U^0; (f; f^0)(u) = \{f(x); x^0 \mid (x; x^0) \in S^{(n+n^0)}; x \in f(u); x^0 \in f^0(u)\}$$

Theorem 15.17 We consider the systems $f : U \rightarrow P^a(S^{(n)}); U \subseteq P^a(S^{(m)})$ and $h : X \rightarrow P^a(S^{(p)}); X \subseteq P^a(S^{(n)})$. Their serial connection is a system if and only if $\exists u \in U; f(u) \cap X \neq \emptyset$. In the case that this condition is fulfilled, we note

$$W = \{u \in U; f(u) \cap X \neq \emptyset\}; g$$

and we have $h \circ f : W \rightarrow P^a(S^{(p)})$,

$$\forall u \in W; (h \circ f)(u) = \{y \in X; y \in h(x) \text{ for } x \in f(u)\}$$

Remark 15.18 Given the system f , its complement Cf is a pseudo-system since

$$\forall u \in S^{(m)}; Cf(u) = S^{(n)} \setminus f(u) \subseteq S^{(n)}$$

If f is a pseudo-system, then Cf can be a system or a pseudo-system.

Theorem 15.19 We consider the systems $f : U \rightarrow P^a(S^{(n)}); g : V \rightarrow P^a(S^{(n)}); U, V \subseteq P^a(S^{(m)})$. Their intersection is a system if and only if

$$\exists u \in U \cap V; f(u) \cap g(u) \neq \emptyset$$

In the case when this condition is fulfilled, we have $f \cap g : W \rightarrow P^a(S^{(n)})$,

$$W = \{u \in U \cap V; f(u) \cap g(u) \neq \emptyset\}; g$$

$$\forall u \in W; (f \cap g)(u) = f(u) \cap g(u)$$

Proof. $W \neq \emptyset$; is the support set of $f \cap g$; we obtain $W \subseteq U \cap V \subseteq S^{(m)}$ and on the other hand we get $\forall u \in W; (f \cap g)(u) \subseteq S^{(n)}$, thus $f \cap g$ is a system. ■

Theorem 15.20 The reunion of the systems f and g is the system $f \cup g : U \cup V \rightarrow P^a(S^{(n)})$ that is defined in the next manner

$$\forall u \in U \cup V; (f \cup g)(u) = f(u) \cup g(u)$$

Proof. $U \neq \emptyset$; and $V \neq \emptyset$; imply $U \cup V \neq \emptyset$; $U \subseteq S^{(m)}$ and $V \subseteq S^{(m)}$ imply $U \cup V \subseteq S^{(m)}$; and $\forall u \in U; f(u) \subseteq S^{(n)}$, $\forall u \in V; g(u) \subseteq S^{(n)}$ imply $\forall u \in U \cup V; (f \cup g)(u) \subseteq S^{(n)}$, thus $f \cup g$ is a system ■

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