ANALELE UNIVERSITATII DIN ORADEA, FASCICOLA MATEMATICA, TOM V, 1995-1996

AN APPLICATION OF THE $N \to B_2$ SEQUENCES' STUDY IN ARITHMETICS

Serban E. Vlad
str. Zimbrului, nr.3, Bl.PB68, Et.2, Ap.11, 3700, Oradea, Romania,
E-mail: serban_e_vlad@yahoo.com

Abstract *A ring isomorphism is given between* $\boldsymbol{Z}$ *and the set of the* $\boldsymbol{N} \to \boldsymbol{B}_2$ *convergent sequences. Its importance is that it shows how the laws from* $\boldsymbol{Z}$ *are handled by the digital devices.*

1.1 $\boldsymbol{B}_2 = \{0,1\}$ is a field together with the laws:

$\oplus$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

table 1

It has the discrete topology, where the $a : \boldsymbol{N} \to \boldsymbol{B}_2, a(n) = a_n, n \in \boldsymbol{N}$ *convergent* sequences ($\boldsymbol{N} = \{0,1,2,...\}$) have the form:

$$a = (a_n) = (a_0, a_1, a_2, ..., \tilde{a}, \tilde{a}, \tilde{a}, ...)$$

and we have written a function by the set of its values. This means that, starting with a certain rank, the sequence becomes constant and the value of the constant is the limit: $a_n \to \tilde{a}$.

We define $S_{0,c} = \{(a_n) \mid a_n \to 0\}, S_{1,c}\{(a_n) \mid a_n \to 1\}, S_c = S_{0,c} \vee S_{1,c}$.

2. Let us define the bijection $L : \boldsymbol{Z} \to S_c$ by:

$$L(z) = \begin{cases} (x_0, x_1, x_2, ..., x_k, 0,0,0,...), \textit{ if } z \geq 0 \textit{ and} \\ \qquad z = x_0 \cdot 2^0 + x_1 \cdot 2^1 + ... + x_k \cdot 2^k \\ (x_0 \oplus 1, x_1 \oplus 1, x_2 \oplus 1, ..., x_k \oplus 1, 1,1,1,...), \textit{ if } z < 0 \textit{ and} \\ \qquad -z - 1 = x_0 \cdot 2^0 + x_1 \cdot 2^1 + ... + x_k \cdot 2^k \end{cases}$$

where $z \in \boldsymbol{Z}$. We have supposed that the relations of compatibility $0_{\boldsymbol{B}_2} = 0_{\boldsymbol{Z}}, 1_{\boldsymbol{B}_2} = 1_{\boldsymbol{Z}}$ are satisfied. We have the association:

$$\begin{pmatrix} \dots \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ \dots \end{pmatrix} \overset{L}{\mapsto} \begin{pmatrix} \dots \\ (0,1,1,1,\dots) \\ (1,1,1,1,\dots) \\ (0,0,0,0,\dots) \\ (1,0,0,0,\dots) \\ (0,1,0,0,\dots) \\ \dots \end{pmatrix}$$

3. L allows the definition of the following laws on S_c, induced from these of $\boldsymbol{Z}$:

a) the sum $S_c \times S_c \ni (x, y) \mapsto z = x + y \in S_c$ by:

$$t_0 = 0$$

$$z_n = x_n \oplus y_n \oplus t_n, t_{n+1} = x_n \cdot y_n \oplus (x_n \oplus y_n) \cdot t_n, n \in \boldsymbol{N}$$

and the sense of the sequence (t_n) is that of transport of a unit to a higher rank.

b) the difference $S_c \times S_c \ni (x, y) \mapsto z = x - y \in S_c$ is:

$$b_0 = 0$$

$$z_n = x_n \oplus y_n \oplus b_n, b_{n+1} = (x_n \oplus y_n) \cdot y_n \oplus (x_n \oplus y_n \oplus 1) \cdot b_n, n \in \boldsymbol{N}$$

c) the inverse $S_c \ni x \mapsto z = -x \in S_c$

$$b_0 = 0$$

$$z_n = x_n \oplus b_n, b_{n+1} = x_n \oplus (x_n \oplus 1) \cdot b_n, n \in \boldsymbol{N}$$

In b), c) the meaning of (b_n) is that of borrowing a unit from a higher rank.

d) In order to give the product $S_c \times S_c \ni (x, y) \mapsto z = x \cdot y \in S_c$, let us note first:

$$e^0 = (1,0,0,0,0,\dots),\ e^n = (\underbrace{0,0,\dots,0}_{n},1,0,0,\dots),\ n \geq 1$$

$$\lambda \cdot (a_0, a_1, a_2, \dots) = \begin{cases} (0,0,0,\dots) & , if\ \lambda = 0 \\ (a_0, a_1, a_2, \dots), & if\ \lambda = 1 \end{cases}$$

and now let us define z by:

d.1) if $y = (0,0,0,\dots)$, then $z = (0,0,0,\dots)$

d.2) if $y = e^0$, then $z = x$

d.3) if $y = e^n, n \geq 1$, then $z = (\underbrace{0,0,\dots,0}_{n}, x_0, x_1, x_2, \dots)$

d.4) if $\{n \mid y_n = 1\}$ is finite and $y = \sum_{n \in \boldsymbol{N}} y_n \cdot e^n$

, then $z = \sum_{n \in \boldsymbol{N}} y_n \cdot (x \cdot e^n)$

d.5) if $y \in S_{1,c}$, then $z = -(x \cdot (-y))$,

where $-y$ has the same form like at d.4)

As $+, \cdot$ are induced from $\boldsymbol{Z}$, we have that S_c is a unitary commutative ring and L is a ring isomorphism.

4. **Example** $L(-5)\cdot L(-3)=(1,1,0,1,1,...)\cdot(1,0,1,1,1,...)=$

$$=-((1,1,0,1,1,...)\cdot(-(1,0,1,1,1,...))=-((1,1,0,1,1,...)\cdot(1,1,0,0,0,...))=$$

$$=-((1,1,0,1,1,...)\cdot(e^0+e^1))=-((1,1,0,1,1,...)\cdot e^0+(1,1,0,1,1,...)\cdot e^1)=$$

$$=-((1,1,0,1,1,...)+(0,1,1,0,1,1,...))=-(1,0,0,0,1,1,...)=(1,1,1,1,0,0,...)=L(15)$$