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ON THE SERIAL CONNECTION OF THE REGULAR ASYNCHRONOUS SYSTEMS

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Abstract The asynchronous systems f are multi-valued functions, representing the non-deterministic models of the asynchronous circuits from the digital electrical engineering. In real time, they map an 'admissible input' function $u : \mathbf{R} \to \{0,1\}^m$ to a set $f(u)$ of 'possible states' $x \in f(u)$, where $x : \mathbf{R} \to \{0,1\}^n$. When f is defined by making use of a 'generator function' $\Phi : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}^n$, the system is called regular. The usual definition of the serial connection of systems as composition of multi-valued functions does not bring the regular systems into regular systems, thus the first issue in this study is to modify in an acceptable manner the definition of the serial connection in a way that matches regularity. This intention was expressed for the first time, without proving the regularity of the serial connection of systems, in the work [3]. Our present purpose is to restate with certain corrections and prove Theorem 45 from that work.

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1. INTRODUCTION

The regular asynchronous systems are the Boolean dynamical systems. They represent the (real time or discrete time) models of the asynchronous circuits from digital electrical engineering. In [3] we have shown that the subsystems of the regular systems are regular, the dual systems of the regular systems are regular, the Cartesian products of the regular systems are regular, the parallel and the serial connections of the regular systems are regular, the intersections and the unions of the regular systems are regular. The result concerning the serial connections given in Theorem 45 from [3] was not proved by that time and our initial purpose was to give its proof. Reconsidering the problem (in a slightly different approach) showed that certain corrections were also necessary. The main result is represented by Theorem 4.1.

2. PRELIMINARIES

Notation 2.1. *We denote with* $\mathbf{B} = \{0,1\}$ *the binary Boole algebra, endowed with the usual laws '$-$' complement, ' $\cdot$ ' intersection, '$\cup$' union and '$\oplus$' exclusive union.*

Definition 2.1. *Let $x : \mathbf{R} \to \mathbf{B}^n$, $y : \mathbf{R} \to \mathbf{B}^p$ be two functions. We define the **Cartesian product** (x, y) of x and y by $(x, y) : \mathbf{R} \to \mathbf{B}^{n+p}$, $\forall i \in \{1, ..., n + p\}, \forall t \in \mathbf{R}$,*

$$(x, y)_i(t) = \begin{cases} x_i(t), \ if \ i \in \{1, ..., n\}, \\ y_i(t), \ if \ i \in \{n + 1, ..., n + p\} \end{cases}.$$

Remark 2.1. *We use to identify $\mathbf{B}^{n+p}$ and $\mathbf{B}^n \times \mathbf{B}^p$. This identification gives us the possibility to write: $\forall t \in \mathbf{R}$,*

$$(x, y)(t) = (x(t), y(t)). \tag{1}$$

Notation 2.2. *We denote by $\chi_H : \mathbf{R} \to \mathbf{B}$ the characteristic function of the set $H \subset \mathbf{R}$: $\forall t \in \mathbf{R}$,*

$$\chi_H(t) = \begin{cases} 1, \ if \ t \in H, \\ 0, \ if \ t \notin H. \end{cases}$$

Notation 2.3. *Seq denotes the set of the sequences $t_0 < t_1 < t_2 < ...$ of real numbers that are unbounded from above. The elements of Seq are usually denoted with $(t_k), (t'_k), ...$*

Definition 2.2. *The function $x : \mathbf{R} \to \mathbf{B}^n$ is called **signal** if $\mu \in \mathbf{B}^n$ and $(t_k) \in Seq$ exist such that*

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... \tag{2}$$

The set of the signals is denoted by $S^{(n)}$.

Definition 2.3. *The function $\rho : \mathbf{R} \to \mathbf{B}^n$ is called **progressive** if $(t_k) \in Seq$ exists such that*

$$\rho(t) = \rho(t_0) \cdot \chi_{\{t_0\}}(t) \oplus \rho(t_1) \cdot \chi_{\{t_1\}}(t) \oplus ... \oplus \rho(t_k) \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{3}$$

and $\forall i \in \{1, ..., n\}$, the set $\{k | k \in \mathbf{N}, \rho_i(t_k) = 1\}$ is infinite. The set of the progressive functions is denoted with P_n.

Theorem 2.1. *a) If $x \in S^{(n)}, y \in S^{(p)}$ then $(x, y) \in S^{(n+p)}$;*
b) If $\rho \in P_n, \widetilde{\rho} \in P_p$ then $(\rho, \widetilde{\rho}) \in P_{n+p}$.

Proof. b) We take arbitrarily $\rho \in P_n, \widetilde{\rho} \in P_p$ for which $\forall t \in \mathbf{R}$,

$$\rho(t) = \rho(t'_0) \cdot \chi_{\{t'_0\}}(t) \oplus \rho(t'_1) \cdot \chi_{\{t'_1\}}(t) \oplus ... \oplus \rho(t'_k) \cdot \chi_{\{t'_k\}}(t) \oplus ... \tag{4}$$

$$\widetilde{\rho}(t) = \widetilde{\rho}(t''_0) \cdot \chi_{\{t''_0\}}(t) \oplus \widetilde{\rho}(t''_1) \cdot \chi_{\{t''_1\}}(t) \oplus ... \oplus \widetilde{\rho}(t''_k) \cdot \chi_{\{t''_k\}}(t) \oplus ... \tag{5}$$

with $(t'_k), (t''_k) \in Seq$. We denote by $(t_k) \in Seq$ the sequence obtained by indexing increasingly the elements of the set $\{t'_k | k \in \mathbf{N}\} \cup \{t''_k | k \in \mathbf{N}\}$. Equations (4), (5) may be rewritten under the form

$$\rho(t) = \rho(t_0) \cdot \chi_{\{t_0\}}(t) \oplus \rho(t_1) \cdot \chi_{\{t_1\}}(t) \oplus ... \oplus \rho(t_k) \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{6}$$

$$\widetilde{\rho}(t) = \widetilde{\rho}(t_0) \cdot \chi_{\{t_0\}}(t) \oplus \widetilde{\rho}(t_1) \cdot \chi_{\{t_1\}}(t) \oplus ... \oplus \widetilde{\rho}(t_k) \cdot \chi_{\{t_k\}}(t) \oplus ... \quad (7)$$

and we get

$$(\rho,\widetilde{\rho})(t) = (\rho(t_0),\widetilde{\rho}(t_0)) \cdot \chi_{\{t_0\}}(t) \oplus (\rho(t_1),\widetilde{\rho}(t_1)) \cdot \chi_{\{t_1\}}(t) \oplus ... \quad (8)$$

$$... \oplus (\rho(t_k),\widetilde{\rho}(t_k)) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

The sets

$$\{k|k \in \mathbf{N}, (\rho,\widetilde{\rho})_i(t_k) = 1\} = \{k|k \in \mathbf{N}, \rho_i(t_k) = 1\}, i = \overline{1,n},$$

$$\{k|k \in \mathbf{N}, (\rho,\widetilde{\rho})_i(t_k) = 1\} = \{k|k \in \mathbf{N}, \widetilde{\rho}_i(t_k) = 1\}, i = \overline{n+1,n+p}$$

are infinite. We conclude that $(\rho,\widetilde{\rho}) \in P_{n+p}$. ∎

3. REGULAR SYSTEMS

Notation 3.1. *$P^*(H)$ is the notation of the non-empty subsets of H. In this paper $H \in \{\mathbf{B}^n, S^{(n)}, P_n\}$.*

Definition 3.1. *A function $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ is called (**asynchronous**) **system**. Any $u \in U$ is called (**admissible**) **input** and any $x \in f(u)$ is called (**possible**) **state**.*

Definition 3.2. *We consider the function $\Phi : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$. For $\nu \in \mathbf{B}^n$, we define $\Phi^\nu : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$ by $\forall \mu \in \mathbf{B}^n, \forall \lambda \in \mathbf{B}^m$,*

$$\Phi^\nu(\mu,\lambda) = (\overline{\nu_1} \cdot \mu_1 \oplus \nu_1 \cdot \Phi_1(\mu,\lambda), ..., \overline{\nu_n} \cdot \mu_n \oplus \nu_n \cdot \Phi_n(\mu,\lambda)).$$

Definition 3.3. *Let be $\mu \in \mathbf{B}^n, U \in P^*(S^{(m)}), u \in U$ and $\rho \in P_n$,*

$$u(t) = \lambda \cdot \chi_{(-\infty,t'_0)}(t) \oplus u(t'_0) \cdot \chi_{[t'_0,t'_1)}(t) \oplus ... \oplus u(t'_k) \cdot \chi_{[t'_k,t'_{k+1})}(t) \oplus ...$$

$$\rho(t) = \rho(t''_0) \cdot \chi_{\{t''_0\}}(t) \oplus \rho(t''_1) \cdot \chi_{\{t''_1\}}(t) \oplus ... \oplus \rho(t''_k) \cdot \chi_{\{t''_k\}}(t) \oplus ...$$

*with $\lambda \in \mathbf{B}^m$ and $(t'_k),(t''_k) \in Seq$. The **orbit** $\Phi^\rho(\mu,u,\cdot) \in S^{(n)}$ is defined like this: we denote by $(t_k) \in Seq$ the elements of the set $\{t'_k|k \in \mathbf{N}\} \cup \{t''_k|k \in \mathbf{N}\}$ indexed increasingly, then $\forall t \in \mathbf{R}$,*

$$\Phi^\rho(\mu,u,t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \omega_0 \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \omega_k \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

where

$$\omega_0 = \Phi^{\rho(t_0)}(\mu,u(t_0)),$$

$$\omega_{k+1} = \Phi^{\rho(t_{k+1})}(\omega_k,u(t_{k+1})), k \in \mathbf{N}.$$

Definition 3.4. *f is called **regular asynchronous system** if Φ and the functions $i_f : U \to P^*(\mathbf{B}^n), \pi_f : \Delta_f \to P^*(P_n)$ exist,*

$$\Delta_f = \{(\mu,u)|u \in U, \mu \in i_f(u)\}$$

such that $\forall u \in U$,

$$f(u) = \{\Phi^{\rho}(\mu, u, \cdot) | \mu \in i_f(u), \rho \in \pi_f(\mu, u)\}.$$

The functions i_f, π_f *are called the* ***initial state function****, respectively the* ***computation function*** *of* f. Φ *is called the* ***generator function*** *of* f*; we also say that* f ***is generated*** *by* Φ.

4. THE SERIAL CONNECTION OF THE REGULAR SYSTEMS

Remark 4.1. *Let be the systems* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *and* $h : X \to P^*(S^{(p)}), X \in P^*(S^{(n)})$ *such that* $\forall u \in U, f(u) \subset X$. *In general the serial connection* $h \circ f$ *of* h *and* f *is defined by* $h \circ f : U \to P^*(S^{(p)}), \forall u \in U, (h \circ f)(u) = \bigcup\limits_{x \in f(u)} h(x) = \{y | x \in f(u), y \in h(x)\}$ *(the composition of the multi-valued functions). This definition does not match regularity, thus we are forced to adopt the following definition of the serial connection, that is still acceptable.*

Definition 4.1. *We define the* ***serial connection*** $h * f$ *of* h *and* f *by* $h * f : U \to P^*(S^{(n+p)})$, $\forall u \in U$,

$$(h * f)(u) = \{(x, y) | x \in f(u), y \in h(x)\}. \tag{9}$$

Definition 4.2. *For the functions* $\Phi : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$ *and* $\Psi : \mathbf{B}^p \times \mathbf{B}^n \to \mathbf{B}^p$ *we define* $\Psi * \Phi : \mathbf{B}^{n+p} \times \mathbf{B}^m \to \mathbf{B}^{n+p}$ *by* $\forall((\mu, \delta), \lambda) \in (\mathbf{B}^n \times \mathbf{B}^p) \times \mathbf{B}^m$,

$$(\Psi * \Phi)((\mu, \delta), \lambda) = (\Phi(\mu, \lambda), \Psi(\delta, \Phi(\mu, \lambda))). \tag{10}$$

In the previous equation we have identified $\mathbf{B}^{n+p}$ *with* $\mathbf{B}^n \times \mathbf{B}^p$.

Definition 4.3. *For any* $\nu \in \mathbf{B}^n, \widetilde{\nu} \in \mathbf{B}^p$ *and* Φ, Ψ *like previously, we define* $(\Psi * \Phi)^{(\nu, \widetilde{\nu})} : \mathbf{B}^{n+p} \times \mathbf{B}^m \to \mathbf{B}^{n+p}$ *in the following manner:*

$$(\Psi * \Phi)^{(\nu, \widetilde{\nu})} = \Psi^{\widetilde{\nu}} * \Phi^{\nu}. \tag{11}$$

Lemma 4.1. *We presume that* $\Phi : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$, $\Psi : \mathbf{B}^p \times \mathbf{B}^n \to \mathbf{B}^p$ *as well as* $\mu \in \mathbf{B}^n, \delta \in \mathbf{B}^p, u \in U, \rho \in P_n, \widetilde{\rho} \in P_p$ *are given. The following formula is true:* $\forall t \in \mathbf{R}$,

$$(\Phi^{\rho}(\mu, u, t), \Psi^{\widetilde{\rho}}(\delta, \Phi^{\rho}(\mu, u, \cdot), t)) = (\Psi * \Phi)^{(\rho, \widetilde{\rho})}((\mu, \delta), u, t).$$

Proof. Let be $u \in U$,

$$u(t) = u(-\infty + 0) \cdot \chi_{(-\infty, t'_0)}(t) \oplus u(t'_0) \cdot \chi_{[t'_0, t'_1)}(t) \oplus ... \tag{12}$$

$$... \oplus u(t'_k) \cdot \chi_{[t'_k, t'_{k+1})}(t) \oplus ...$$

together with the functions $\rho \in P_n, \widetilde{\rho} \in P_p$,

$$\rho(t) = \rho(t''_0) \cdot \chi_{\{t''_0\}}(t) \oplus \rho(t''_1) \cdot \chi_{\{t''_1\}}(t) \oplus ... \oplus \rho(t''_k) \cdot \chi_{\{t''_k\}}(t) \oplus ... \tag{13}$$

$$\widetilde{\rho}(t) = \widetilde{\rho}(t'''_0) \cdot \chi_{\{t'''_0\}}(t) \oplus \widetilde{\rho}(t'''_1) \cdot \chi_{\{t'''_1\}}(t) \oplus ... \oplus \widetilde{\rho}(t'''_k) \cdot \chi_{\{t'''_k\}}(t) \oplus ... \tag{14}$$

where $(t'_k), (t''_k), (t'''_k) \in Seq$. We denote with $(t_k) \in Seq$ the sequence that is obtained by indexing increasingly the elements of the set $\{t'_k | k \in \mathbf{N}\} \cup \{t''_k | k \in \mathbf{N}\} \cup \{t'''_k | k \in \mathbf{N}\}$, for which (12), (13), (14) may be rewriten under the form

$$u(t) = u(-\infty + 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus u(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \tag{15}$$

$$... \oplus u(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

$$\rho(t) = \rho(t_0) \cdot \chi_{\{t_0\}}(t) \oplus \rho(t_1) \cdot \chi_{\{t_1\}}(t) \oplus ... \oplus \rho(t_k) \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{16}$$

$$\widetilde{\rho}(t) = \widetilde{\rho}(t_0) \cdot \chi_{\{t_0\}}(t) \oplus \widetilde{\rho}(t_1) \cdot \chi_{\{t_1\}}(t) \oplus ... \oplus \widetilde{\rho}(t_k) \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{17}$$

We have

$$\Phi^{\rho}(\mu, u, t) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \omega_0 \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus \omega_k \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ... \tag{18}$$

where

$$\omega_0 = \Phi^{\rho(t_0)}(\mu, u(t_0)), \tag{19}$$

$$\omega_{k+1} = \Phi^{\rho(t_{k+1})}(\omega_k, u(t_{k+1})), k \in \mathbf{N} \tag{20}$$

and furthermore

$$\Psi^{\widetilde{\rho}}(\delta, \Phi^{\rho}(\mu, u, \cdot), t) = \delta \cdot \chi_{(-\infty, t_0)}(t) \oplus \gamma_0 \cdot \chi_{[t_0, t_1)}(t) \oplus ... \tag{21}$$

$$... \oplus \gamma_k \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

where

$$\gamma_0 = \Psi^{\widetilde{\rho}(t_0)}(\delta, \omega_0), \tag{22}$$

$$\gamma_{k+1} = \Psi^{\widetilde{\rho}(t_{k+1})}(\gamma_k, \omega_{k+1}), k \in \mathbf{N}. \tag{23}$$

We conclude from (18),(21) that

$$(\Phi^{\rho}(\mu, u, t), \Psi^{\widetilde{\rho}}(\delta, \Phi^{\rho}(\mu, u, \cdot), t)) = (\mu, \delta) \cdot \chi_{(-\infty, t_0)}(t) \oplus \tag{24}$$

$$\oplus (\omega_0, \gamma_0) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus (\omega_k, \gamma_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

On the other hand

$$(\rho, \widetilde{\rho})(t) = (\rho(t_0), \widetilde{\rho}(t_0)) \cdot \chi_{\{t_0\}}(t) \oplus (\rho(t_1), \widetilde{\rho}(t_1)) \cdot \chi_{\{t_1\}}(t) \oplus ... \tag{25}$$

$$... \oplus (\rho(t_k), \widetilde{\rho}(t_k)) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

thus

$$(\Psi * \Phi)^{(\rho,\widetilde{\rho})}((\mu,\delta),u,t) = (\mu,\delta)\cdot\chi_{(-\infty,t_0)}(t)\oplus\theta_0\cdot\chi_{[t_0,t_1)}(t)\oplus ... \qquad (26)$$
$$... \oplus\theta_k\cdot\chi_{[t_k,t_{k+1})}(t)\oplus ...$$

is true with

$$\theta_0 = (\Psi * \Phi)^{(\rho,\widetilde{\rho})(t_0)}((\mu,\delta),u(t_0)), \qquad (27)$$

$$\theta_{k+1} = (\Psi * \Phi)^{(\rho,\widetilde{\rho})(t_{k+1})}(\theta_k,u(t_{k+1})), k\in\mathbf{N}. \qquad (28)$$

We prove by induction on k that

$$\theta_k = (\omega_k,\gamma_k), k\in\mathbf{N}. \qquad (29)$$

We have

$$\theta_0 \overset{(27)}{=} (\Psi * \Phi)^{(\rho,\widetilde{\rho})(t_0)}((\mu,\delta),u(t_0)) \overset{(1)}{=} (\Psi * \Phi)^{(\rho(t_0),\widetilde{\rho}(t_0))}((\mu,\delta),u(t_0))$$
$$\overset{(11)}{=} (\Psi^{\widetilde{\rho}(t_0)} * \Phi^{\rho(t_0)})((\mu,\delta),u(t_0))$$
$$\overset{(10)}{=} (\Phi^{\rho(t_0)}(\mu,u(t_0)),\Psi^{\widetilde{\rho}(t_0)}(\delta,\Phi^{\rho(t_0)}(\mu,u(t_0))))$$
$$\overset{(19)}{=} (\omega_0,\Psi^{\widetilde{\rho}(t_0)}(\delta,\omega_0)) \overset{(22)}{=} (\omega_0,\gamma_0)$$

thus (29) is true for $k=0$. We presume that it is true for k and we prove it for $k+1$:

$$\theta_{k+1} \overset{(28)}{=} (\Psi * \Phi)^{(\rho,\widetilde{\rho})(t_{k+1})}(\theta_k,u(t_{k+1})) \overset{(1)}{=} (\Psi * \Phi)^{(\rho(t_{k+1}),\widetilde{\rho}(t_{k+1}))}(\theta_k,u(t_{k+1}))$$
$$\overset{(11)}{=} (\Psi^{\widetilde{\rho}(t_{k+1})} * \Phi^{\rho(t_{k+1})})(\theta_k,u(t_{k+1})) \overset{hyp}{=} (\Psi^{\widetilde{\rho}(t_{k+1})} * \Phi^{\rho(t_{k+1})})((\omega_k,\gamma_k),u(t_{k+1}))$$
$$\overset{(10)}{=} (\Phi^{\rho(t_{k+1})}(\omega_k,u(t_{k+1})),\Psi^{\widetilde{\rho}(t_{k+1})}(\gamma_k,\Phi^{\rho(t_{k+1})}(\omega_k,u(t_{k+1}))))$$
$$\overset{(20)}{=} (\omega_{k+1},\Psi^{\widetilde{\rho}(t_{k+1})}(\gamma_k,\omega_{k+1})) \overset{(23)}{=} (\omega_{k+1},\gamma_{k+1}).$$

Equation (29) is proved and its truth shows, from (24) and (26), the validity of the statement of the Lemma. ∎

Definition 4.4. *The functions* $i_f : U \to P^*(\mathbf{B}^n)$ *and* $\pi_f : \Delta_f \to P^*(P_n)$ *are given,*

$$\Delta_f = \{(\mu,u)|u\in U,\mu\in i_f(u)\}$$

such that $\forall u\in U$,

$$f(u) = \{\Phi^\rho(\mu,u,\cdot)|\mu\in i_f(u),\rho\in\pi_f(\mu,u)\} \qquad (30)$$

and similarly the functions $i_h : X \to P^*(\mathbf{B}^p)$ *and* $\pi_h : \Delta_h \to P^*(P_p)$ *are given,*

$$\Delta_h = \{(\delta,x)|x\in X,\delta\in i_h(x)\}$$

such that $\forall x \in X$,

$$h(x) = \{\Psi^{\widetilde{\rho}}(\delta, x, \cdot)|\delta \in i_h(x), \widetilde{\rho} \in \pi_h(\delta, x)\}. \tag{31}$$

We presume that $\forall u \in U, f(u) \subset X$. *We define* $i : U \to P^*(\mathbf{B}^{n+p})$ *by* $\forall u \in U$,

$$i(u) = \{(\mu, \delta)|\mu \in i_f(u), \exists\rho' \in \pi_f(\mu, u), \delta \in i_h(\Phi^{\rho'}(\mu, u, \cdot))\} \tag{32}$$

and $\pi : \Delta \to P^*(P_{n+p})$ *respectively by*

$$\Delta = \{((\mu, \delta), u)|u \in U, \mu \in i_f(u), \exists\rho \in \pi_f(\mu, u), \delta \in i_h(\Phi^{\rho}(\mu, u, \cdot))\},$$

$\forall((\mu, \delta), u) \in \Delta$,

$$\pi((\mu, \delta), u) = \tag{33}$$

$$= \{(\rho, \widetilde{\rho})|\rho \in \pi_f(\mu, u), \delta \in i_h(\Phi^{\rho}(\mu, u, \cdot)), \widetilde{\rho} \in \pi_h(\delta, \Phi^{\rho}(\mu, u, \cdot))\}.$$

Lemma 4.2. *The following equality holds for* $u \in U$:

$$\{(\Psi * \Phi)^{(\rho, \widetilde{\rho})}((\mu, \delta), u, \cdot)|\mu \in i_f(u), \rho \in \pi_f(\mu, u),$$

$$\delta \in i_h(\Phi^{\rho}(\mu, u, \cdot)), \widetilde{\rho} \in \pi_h(\delta, \Phi^{\rho}(\mu, u, \cdot))\}$$

$$= \{(\Psi * \Phi)^{(\rho, \widetilde{\rho})}((\mu, \delta), u, \cdot)|(\mu, \delta) \in i(u), (\rho, \widetilde{\rho}) \in \pi((\mu, \delta), u)\}.$$

Proof. We denote for $u \in U$

$$A = \{(\Psi * \Phi)^{(\rho, \widetilde{\rho})}((\mu, \delta), u, \cdot)|\mu \in i_f(u), \rho \in \pi_f(\mu, u),$$

$$\delta \in i_h(\Phi^{\rho}(\mu, u, \cdot)), \widetilde{\rho} \in \pi_h(\delta, \Phi^{\rho}(\mu, u, \cdot))\},$$

$$B = \{(\Psi * \Phi)^{(\rho, \widetilde{\rho})}((\mu, \delta), u, \cdot)|(\mu, \delta) \in i(u), (\rho, \widetilde{\rho}) \in \pi((\mu, \delta), u)\}.$$

Let $(\Psi * \Phi)^{(\rho, \widetilde{\rho})}((\mu, \delta), u, \cdot) \in A$ be arbitrary, where

$$\mu \in i_f(u), \rho \in \pi_f(\mu, u), \delta \in i_h(\Phi^{\rho}(\mu, u, \cdot)), \widetilde{\rho} \in \pi_h(\delta, \Phi^{\rho}(\mu, u, \cdot)). \tag{34}$$

We infer that

$$\mu \in i_f(u), \exists\rho' \in \pi_f(\mu, u), \delta \in i_h(\Phi^{\rho'}(\mu, u, \cdot)), \tag{35}$$

$$\rho \in \pi_f(\mu, u), \delta \in i_h(\Phi^{\rho}(\mu, u, \cdot)), \widetilde{\rho} \in \pi_h(\delta, \Phi^{\rho}(\mu, u, \cdot))$$

holds, from (32), (33) we get

$$(\mu, \delta) \in i(u), (\rho, \widetilde{\rho}) \in \pi((\mu, \delta), u), \tag{36}$$

thus $(\Psi * \Phi)^{(\rho, \widetilde{\rho})}((\mu, \delta), u, \cdot) \in B$ and finally $A \subset B$.

Conversely, let $(\Psi * \Phi)^{(\rho, \widetilde{\rho})}((\mu, \delta), u, \cdot) \in B$ be arbitrary, with (36) fulfilled, wherefrom we get that (35) is true. Then (34) holds, meaning that $B \subset A$.

The statement of the Theorem is proved. ∎

Theorem 4.1. *We have that* $\forall u \in U$,

$$(h * f)(u) = \{(\Psi * \Phi)^{(\rho,\widetilde{\rho})}((\mu,\delta),u,\cdot)|(\mu,\delta) \in i(u), (\rho,\widetilde{\rho}) \in \pi((\mu,\delta),u)\},$$

i.e. $h * f$ *is regular generated by* $\Psi * \Phi$, $i = i_{h*f}$ *and* $\pi = \pi_{h*f}$.

Proof. Let $u \in U$ be arbitrary. We infer:

$$(h * f)(u) \overset{(9)}{=} \{(x,y)|x \in f(u), y \in h(x)\}$$

$$\overset{(30),(31)}{=} \{(\Phi^{\rho}(\mu,u,\cdot), \Psi^{\widetilde{\rho}}(\delta,\Phi^{\rho}(\mu,u,\cdot),\cdot)|\mu \in i_f(u), \rho \in \pi_f(\mu,u),$$
$$\delta \in i_h(\Phi^{\rho}(\mu,u,\cdot)), \widetilde{\rho} \in \pi_h(\delta,\Phi^{\rho}(\mu,u,\cdot))\}$$

$$\overset{Lemma\ 4.1}{=} \{(\Psi * \Phi)^{(\rho,\widetilde{\rho})}((\mu,\delta),u,\cdot)|\mu \in i_f(u), \rho \in \pi_f(\mu,u),$$
$$\delta \in i_h(\Phi^{\rho}(\mu,u,\cdot)), \widetilde{\rho} \in \pi_h(\delta,\Phi^{\rho}(\mu,u,\cdot))\}$$

$$\overset{Lemma\ 4.2}{=} \{(\Psi * \Phi)^{(\rho,\widetilde{\rho})}((\mu,\delta),u,\cdot)|(\mu,\delta) \in i(u), (\rho,\widetilde{\rho}) \in \pi((\mu,\delta),u)\}.$$

∎

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