

Universal Regular Autonomous Asynchronous Systems: Fixed Points, Equivalencies and Dynamical Bifurcations

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Abstract

The asynchronous systems are the non-deterministic models of the asynchronous circuits from the digital electrical engineering. In the autonomous version, such a system is a set of functions $x : \mathbf{R} \rightarrow \{0, 1\}^n$ called states ($\mathbf{R}$ is the time set). If an autonomous asynchronous system is defined by making use of a so called generator function $\Phi : \{0, 1\}^n \rightarrow \{0, 1\}^n$, then it is called regular. The regular autonomous asynchronous systems compute in real time the iterates of Φ when these are not made, in general, on all the coordinates $\Phi_1, \dots, \Phi_n$ simultaneously. The property of universality means the greatest in the sense of the inclusion.

The purpose of the paper is that of defining and of characterizing the fixed points, the equivalencies and the dynamical bifurcations of the universal regular autonomous asynchronous systems. We use analogies with the dynamical systems theory.

Key words: asynchronous system, fixed point, dynamical bifurcation

AMS subject classifications: 94C10, 93A30, 47H10, 37G99

1 Introduction

Switching theory, more precisely: what we mean by switching theory, has been practiced in the 50's and the 60's by many mathematicians, in dialogue with engineers. The last book from this series was published by Moisil¹ in 1969. After 1970, the theory of modeling the asynchronous circuits from the digital electrical engineering has developed in a manner suggesting that the main interest of the researchers is to keep away their

¹Gr. C. Moisil, The algebraic theory of switching circuits, Pergamon Press and Editura tehnica, Bucharest, 1969

works from publication. In this context, we have started some years ago the construction of a theory of modeling the asynchronous circuits under the name of asynchronous systems theory. A part of this theory, related with the universal regular autonomous asynchronous systems is presented in this paper. The bibliography that we indicate consists in works on dynamical systems (written as usual on real numbers, we use binary numbers here) that create analogies. They are not relevant to the readers that are familiar with the concepts of orbit, nullclin, dynamical bifurcation etc, except for showing the source of inspiration of the construction. The paper is obviously self-contained.

2 Preliminaries

Definition 1 We denote by $\mathbf{B} = \{0, 1\}$ the **binary Boole algebra**, endowed with the discrete topology and with the usual laws.

Definition 2 Let be the Boolean function $\Phi : \mathbf{B}^n \rightarrow \mathbf{B}^n$, $\Phi = (\Phi_1, \dots, \Phi_n)$ and $\nu \in \mathbf{B}^n$, $\nu = (\nu_1, \dots, \nu_n)$. We define $\Phi^\nu : \mathbf{B}^n \rightarrow \mathbf{B}^n$ by $\forall \mu \in \mathbf{B}^n$,

$$\Phi^\nu(\mu) = (\overline{\nu_1} \cdot \mu_1 \oplus \nu_1 \cdot \Phi_1(\mu), \dots, \overline{\nu_n} \cdot \mu_n \oplus \nu_n \cdot \Phi_n(\mu)).$$

Remark 3 Φ^ν represents the function resulting from Φ when this one is not computed, in general, on all the coordinates Φ_i , $i = \overline{1, n}$: if $\nu_i = 0$, then Φ_i is not computed, $\Phi_i^\nu(\mu) = \mu_i$ and if $\nu_i = 1$, then Φ_i is computed, $\Phi_i^\nu(\mu) = \Phi_i(\mu)$.

Definition 4 Let be the sequence $\alpha^0, \alpha^1, \dots, \alpha^k, \dots \in \mathbf{B}^n$. The functions $\Phi^{\alpha^0 \alpha^1 \dots \alpha^k} : \mathbf{B}^n \rightarrow \mathbf{B}^n$ are defined iteratively by $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}$,

$$\Phi^{\alpha^0 \alpha^1 \dots \alpha^k \alpha^{k+1}}(\mu) = \Phi^{\alpha^{k+1}}(\Phi^{\alpha^0 \alpha^1 \dots \alpha^k}(\mu)).$$

Definition 5 The sequence $\alpha^0, \alpha^1, \dots, \alpha^k, \dots \in \mathbf{B}^n$ is called **progressive** if

$$\forall i \in \{1, \dots, n\}, \text{ the set } \{k | k \in \mathbf{N}, \alpha_i^k = 1\} \text{ is infinite.}$$

The set of the progressive sequences is denoted by Π_n .

Remark 6 Let be $\mu \in \mathbf{B}^n$. When $\alpha = \alpha^0, \alpha^1, \dots, \alpha^k, \dots$ is progressive, each coordinate Φ_i , $i = \overline{1, n}$ is computed infinitely many times in the sequence $\Phi^{\alpha^0 \alpha^1 \dots \alpha^k}(\mu)$, $k \in \mathbf{N}$.

Definition 7 The **initial value**, denoted by $x(-\infty + 0)$ or $\lim_{t \rightarrow -\infty} x(t) \in \mathbf{B}^n$ and the **final value**, denoted by $x(\infty - 0)$ or $\lim_{t \rightarrow \infty} x(t) \in \mathbf{B}^n$ of the function $x : \mathbf{R} \rightarrow \mathbf{B}^n$ are defined by

$$\exists t' \in \mathbf{R}, \forall t < t', x(t) = x(-\infty + 0),$$

$$\exists t' \in \mathbf{R}, \forall t > t', x(t) = x(\infty - 0).$$

Definition 8 The function $x : \mathbf{R} \rightarrow \mathbf{B}^n$ is called *(pseudo)periodical with the period $T_0 > 0$* if

- a) $\lim_{t \rightarrow \infty} x(t)$ does not exist and
- b) $\exists t' \in \mathbf{R}, \forall t \geq t', x(t) = x(t + T_0)$.

Definition 9 The *characteristic function* $\chi_A : \mathbf{R} \rightarrow \mathbf{B}$ of the set $A \subset \mathbf{R}$ is defined in the following way:

$$\chi_A(t) = \begin{cases} 1, & \text{if } t \in A \\ 0, & \text{otherwise} \end{cases}.$$

Notation 10 We denote by *Seq* the set of the real sequences $t_0 < t_1 < \dots < t_k < \dots$ which are unbounded from above.

Remark 11 The sequences $(t_k) \in \text{Seq}$ act as time sets. At this level of generality of the exposure, a double uncertainty exists in the real time iterative computations of the function $\Phi : \mathbf{B}^n \rightarrow \mathbf{B}^n$: we do not know precisely neither the coordinates Φ_i of Φ that are computed, nor when the computation happens. This uncertainty implies the non-determinism of the model and its origin consists in structural fluctuations in the fabrication process, the variations in ambiental temperature and the power supply etc.

Definition 12 A *signal* (or *n-signal*) is a function $x : \mathbf{R} \rightarrow \mathbf{B}^n$ of the form

$$\begin{aligned} x(t) = & x(-\infty + 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus \dots \\ & \dots \oplus x(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots \end{aligned} \quad (1)$$

with $(t_k) \in \text{Seq}$. The set of the signals is denoted by $S^{(n)}$.

Remark 13 The signals $x \in S^{(n)}$ model the electrical signals from the digital electrical engineering. They have by definition initial values and they avoid 'Dirichlet type' properties (called Zeno properties by the engineers) such as

$$\begin{aligned} \exists t \in \mathbf{R}, \forall \varepsilon > 0, \exists t' \in (t - \varepsilon, t), \exists t'' \in (t - \varepsilon, t), x(t') \neq x(t''), \\ \exists t \in \mathbf{R}, \forall \varepsilon > 0, \exists t' \in (t, t + \varepsilon), \exists t'' \in (t, t + \varepsilon), x(t') \neq x(t'') \end{aligned}$$

because these properties cannot characterize the inertial devices.

We can interpret now Definition 8 of (pseudo)periodicity in the situation when $x \in S^{(n)}$. If at b) we would have had $\forall t \in \mathbf{R}, x(t) = x(t + T_0)$, then the existence of $x(-\infty + 0)$ implies that x is constant. Similarly, if a) would be false, then x would be constant. In other words Definition 8 was formulated in a way that makes us work with non-constant functions, a request of non-triviality.

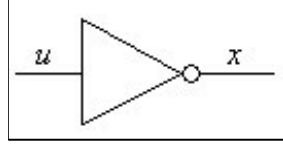


Figure 1: Circuit with the logical gate NOT

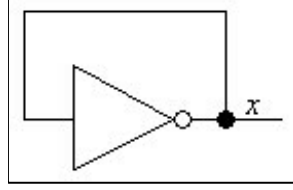


Figure 2: Circuit with feedback with the logical gate NOT

Notation 14 We denote by P^* the set of the non-empty subsets of a set.

Definition 15 The **autonomous asynchronous systems** are the non-empty sets $X \in P^*(S^{(n)})$.

Example 16 We give the following simple example that shows how the autonomous asynchronous systems model the asynchronous circuits. In Figure 1 we have drawn the (logical) gate NOT with the input $u \in S^{(1)}$ and the state (the output) $x \in S^{(1)}$. For $\lambda \in \mathbf{B}$ and

$$u(t) = \lambda,$$

the state x represents the computation of the negation of u and it is of the form

$$\begin{aligned} x(t) &= \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \bar{\lambda} \cdot \chi_{[t_0, t_1)}(t) \oplus \bar{\lambda} \cdot \chi_{[t_1, t_2)}(t) \oplus \dots \oplus \bar{\lambda} \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots \\ &= \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \bar{\lambda} \cdot \chi_{[t_0, \infty)}(t), \end{aligned}$$

where $\mu \in \mathbf{B}$ is the initial value of x and $(t_k) \in \text{Seq}$ is arbitrary. As we can see, x depends on t_0, μ, λ only and it is independent on $t_1, t_2, \dots$

In Figure 2, we have

$$\begin{aligned} x(t) &= \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \bar{\mu} \cdot \chi_{[t_0, t_1)}(t) \oplus \mu \cdot \chi_{[t_1, t_2)}(t) \oplus \dots \\ &\quad \oplus \bar{\mu} \cdot \chi_{[t_{2k}, t_{2k+1})}(t) \oplus \mu \cdot \chi_{[t_{2k+1}, t_{2k+2})}(t) \oplus \dots \end{aligned}$$

thus this circuit is modeled by the autonomous asynchronous system

$$X = \{\mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \bar{\mu} \cdot \chi_{[t_0, t_1)}(t) \oplus \mu \cdot \chi_{[t_1, t_2)}(t) \oplus \dots \\ \oplus \bar{\mu} \cdot \chi_{[t_{2k}, t_{2k+1})}(t) \oplus \mu \cdot \chi_{[t_{2k+1}, t_{2k+2})}(t) \oplus \dots | \mu \in \mathbf{B}, (t_k) \in Seq\} \in P^*(S^{(1)}).$$

Definition 17 The *progressive functions* $\rho : \mathbf{R} \rightarrow \mathbf{B}^n$ are by definition the functions

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus \alpha^1 \cdot \chi_{\{t_1\}}(t) \oplus \dots \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus \dots \quad (2)$$

where $(t_k) \in Seq$ and $\alpha^0, \alpha^1, \dots, \alpha^k, \dots \in \Pi_n$. The set of the progressive functions is denoted by P_n .

Definition 18 For $\Phi : \mathbf{B}^n \rightarrow \mathbf{B}^n$ and $\rho \in P_n$ like at (2), we define $\Phi^\rho : \mathbf{B}^n \times \mathbf{R} \rightarrow \mathbf{B}^n$ by $\forall \mu \in \mathbf{B}^n, \forall t \in \mathbf{R}$,

$$\Phi^\rho(\mu, t) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0, t_1)}(t) \oplus \dots \oplus \Phi^{\alpha^0 \dots \alpha^k}(\mu) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots$$

Remark 19 The previous equation reminds the iterations of a discrete time real dynamical system. The time is not exactly discrete in it, but some sort of intermediate situation occurs between the discrete and the real time; on the other hand the iterations of Φ do not happen in general on all the coordinates (synchronicity), but on some coordinates only, such that any coordinate Φ_i is computed infinitely many times, $i = \overline{1, n}$ (asynchronicity) when $t \in \mathbf{R}$. This is the meaning of the progress property, giving the so called 'unbounded delay model' of computation of the Boolean functions.

3 Discrete time

Notation 20 We denote by

$$\mathbf{N}_- = \mathbf{N} \cup \{-1\}$$

the discrete time set.

Definition 21 Let be $\Phi : \mathbf{B}^n \rightarrow \mathbf{B}^n$ and $\alpha \in \Pi_n, \alpha = \alpha^0, \dots, \alpha^k, \dots$. We define the function $\widehat{\Phi}^\alpha : \mathbf{B}^n \times \mathbf{N}_- \rightarrow \mathbf{B}^n$ by $\forall (\mu, k) \in \mathbf{B}^n \times \mathbf{N}_-$,

$$\widehat{\Phi}^\alpha(\mu, k) = \begin{cases} \mu, & k = -1, \\ \Phi^{\alpha^0 \dots \alpha^k}(\mu), & k \geq 0 \end{cases}.$$

Notation 22 Let us denote

$$\widehat{\Pi}_n = \{\alpha | \alpha \in \Pi_n, \forall k \in \mathbf{N}, \alpha^k \neq (0, \dots, 0)\}.$$

Definition 23 The equivalence of $\rho, \rho' \in P_n$ is defined by: $\exists(t_k) \in Seq, \exists(t'_k) \in Seq, \exists\alpha \in \widehat{\Pi}_n$ such that (2) and

$$\rho'(t) = \alpha^0 \cdot \chi_{\{t'_0\}}(t) \oplus \alpha^1 \cdot \chi_{\{t'_1\}}(t) \oplus \dots \oplus \alpha^k \cdot \chi_{\{t'_k\}}(t) \oplus \dots$$

are true.

Definition 24 The 'canonical surjection' $s : P_n \rightarrow \widehat{\Pi}_n$ is by definition the function $\forall \rho \in P_n$,

$$s(\rho) = \alpha$$

where $\alpha \in \widehat{\Pi}_n$ is the only sequence such that $(t_k) \in Seq$ exists, making the equation (2) true.

Remark 25 The relation between the continuous and the discrete time is the following: for any $\mu \in \mathbf{B}^n$ and any $\rho \in P_n$, the sequences $\alpha \in \widehat{\Pi}_n$ and $(t_k) \in Seq$ exist making the equation (2) true and we have

$$\begin{aligned} \Phi^\rho(\mu, t) &= \widehat{\Phi}^\alpha(\mu, -1) \cdot \chi_{(-\infty, t_0)}(t) \oplus \widehat{\Phi}^\alpha(\mu, 0) \cdot \chi_{[t_0, t_1)}(t) \oplus \dots \\ &\dots \oplus \widehat{\Phi}^\alpha(\mu, k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots \end{aligned}$$

Equivalent progressive functions $\rho, \rho' \in P_n$ (i.e. $s(\rho) = s(\rho')$) give 'equivalent' functions $\Phi^\rho(\mu, t), \Phi^{\rho'}(\mu, t)$ in the sense that the computations $\widehat{\Phi}^\alpha(\mu, k), k \in \mathbf{N}_-$ are the same $\forall \mu \in \mathbf{B}^n$, but the time flow is piecewise faster or slower in the two situations.

4 Regular autonomous asynchronous systems

Definition 26 The **universal regular autonomous asynchronous system** $\Xi_\Phi \in P^*(S^{(n)})$ that is generated by the function $\Phi : \mathbf{B}^n \rightarrow \mathbf{B}^n$ is defined by

$$\Xi_\Phi = \{\Phi^\rho(\mu, \cdot) \mid \mu \in \mathbf{B}^n, \rho \in P_n\}.$$

Definition 27 An autonomous asynchronous system $X \in P^*(S^{(n)})$ is called **regular**, if Φ exists such that $X \subset \Xi_\Phi$. In this case Φ is called the **generator function**² of X .

Remark 28 In the last two definitions, the attribute 'regular' refers to the existence of a generator function Φ and the attribute 'universal' means maximal relative to the inclusion.

For a regular system, Φ is not unique in general.

²The terminology of 'generator function' is also used in [1], meaning the vector field of a discrete time dynamical system. In [3] the terminology of 'generator' (function) of a dynamical system is mentioned too. Moisiil called Φ 'network function' in a non-autonomous, discrete time context; for Moisiil, 'network' means 'system' or 'circuit'.

Example 29 For any $\mu^0 \in \mathbf{B}^n$ and $\rho^* \in P_n$, the autonomous systems $\{\Phi^{\rho^*}(\mu^0, \cdot)\}$, $\{\Phi^\rho(\mu^0, \cdot) | \rho \in P_n\}$, $\{\Phi^{\rho^*}(\mu, \cdot) | \mu \in \mathbf{B}^n\}$ and Ξ_Φ are regular.

For $\Phi = 1_{\mathbf{B}^n}$, the system $\Xi_{1_{\mathbf{B}^n}} = \{\mu | \mu \in \mathbf{B}^n\} = \mathbf{B}^n$ is regular and we have identified the constant function $x \in S^{(n)}$, $x(t) = \mu$ with the constant $\mu \in \mathbf{B}^n$.

Another example of universal regular autonomous asynchronous system is given by $\Phi = \mu^0$, the constant function, for which $\Xi_{\mu^0} = \{x | x_i = \mu_i \cdot \chi_{(-\infty, t_i)} \oplus \mu_i^0 \cdot \chi_{[t_i, \infty)}, \mu_i \in \mathbf{B}, t_i \in \mathbf{R}, i = \overline{1, n}\}$.

Remark 30 These examples suggest several possibilities of defining the systems $X \subset \Xi_\Phi$ which are not universal. For example by putting appropriate supplementary requests on the functions ρ , one could rediscover the 'bounded delay model' of computation of the Boolean functions. If ρ is fixed, we get the 'fixed delay model' of computation of the Boolean functions.

5 Orbits and state portraits

Definition 31 Let be $\rho \in P_n$. Two things are understood by **orbit**, or (**state**, or **phase**) **trajectory of Ξ_Φ starting at $\mu \in \mathbf{B}^n$** :

- a) the function $\Phi^\rho(\mu, \cdot) : \mathbf{R} \rightarrow \mathbf{B}^n$;
- b) the set $Or_\rho(\mu) = \{\Phi^\rho(\mu, t) | t \in \mathbf{R}\}$ representing the values of the previous function.

Sometimes the function from a) is called the **motion** (or the **dynamic**) of μ through Φ^ρ .

Definition 32 The equivalent properties

$$\exists t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu'$$

and

$$\mu' \in Or_\rho(\mu)$$

are called of **accessibility**; the points $\mu' \in Or_\rho(\mu)$ are said to be **accessible**.

Remark 33 The orbits are the curves in $\mathbf{B}^n$, parametrized by ρ and t . On the other hand $\rho \in P_n$, $t' \in \mathbf{R}$ imply $\rho \cdot \chi_{(t', \infty)} \in P_n$ and we see the truth of the implication

$$\mu' = \Phi^\rho(\mu, t') \implies \forall t \geq t', \Phi^\rho(\mu, t) = \Phi^{\rho \cdot \chi_{(t', \infty)}}(\mu', t).$$

Definition 34 The **state** (or the **phase**) **portrait** of Ξ_Φ is the set of its orbits.

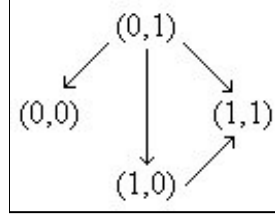


Figure 3: The state portrait of the system from Example 35.

Example 35 The function $\Phi : \mathbf{B}^2 \rightarrow \mathbf{B}^2$ is defined by the following table

(μ_1, μ_2)	$\Phi(\mu_1, \mu_2)$
(0, 0)	(0, 0)
(0, 1)	(1, 0)
(1, 0)	(1, 1)
(1, 1)	(1, 1)

The state portrait of Ξ_Φ is:

$$\begin{aligned}
& \{(0, 1) \cdot \chi_{(-\infty, t_0)} \oplus (0, 0) \cdot \chi_{[t_0, \infty)} | t_0 \in \mathbf{R}\} \cup \\
& \cup \{(0, 1) \cdot \chi_{(-\infty, t_0)} \oplus (1, 0) \cdot \chi_{[t_0, t_1)} \oplus (1, 1) \cdot \chi_{[t_1, \infty)} | t_0, t_1 \in \mathbf{R}, t_0 < t_1\} \cup \\
& \cup \{(0, 1) \cdot \chi_{(-\infty, t_0)} \oplus (1, 1) \cdot \chi_{[t_0, \infty)} | t_0 \in \mathbf{R}\} \cup \\
& \cup \{(1, 0) \cdot \chi_{(-\infty, t_0)} \oplus (1, 1) \cdot \chi_{[t_0, \infty)} | t_0 \in \mathbf{R}\} \cup \{(0, 0)\} \cup \{(1, 1)\}.
\end{aligned}$$

This set is drawn in Figure 3, where the arrows show the increase of time. One might want to put arrows from (0, 0) to itself and from (1, 1) to itself.

6 Nullclins

Definition 36 Let be $\Phi : \mathbf{B}^n \rightarrow \mathbf{B}^n$. For any $i \in \{1, \dots, n\}$, the **nullclins** of Φ are the sets

$$NC_i = \{\mu | \mu \in \mathbf{B}^n, \Phi_i(\mu) = \mu_i\}.$$

If $\mu \in NC_i$, then the coordinate i is said to be **not excited**, or **not enabled**, or **stable** and if $\mu \in \mathbf{B}^n \setminus NC_i$ then it is called **excited**, or **enabled**, or **unstable**.

Remark 37 Sometimes, instead of indicating Φ by a table like previously, we can replace Figure 3 by Figure 4, where we have underlined the unstable coordinates. For example in Figure 4, $(\underline{0}, \underline{1})$ means that $\Phi(0, 1) = (1, 0)$, $(1, \underline{0})$ means that $\Phi(1, 0) = (1, 1)$ etc.

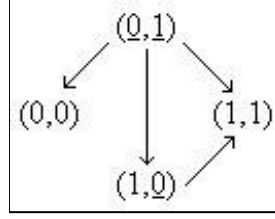


Figure 4: The state portrait of the system from Example 35, version

In fact Figure 4 results uniquely from Figure 3, one could know by looking at Figure 3 which coordinates should be underlined and which should be not. For example the existence of an arrow from $(0,1)$ to $(1,0)$ shows that in $(0,1)$ both coordinates are enabled.

7 Fixed points. Rest position

Definition 38 A point $\mu \in \mathbf{B}^n$ that fulfills $\Phi(\mu) = \mu$ is called a **fixed point** (an **equilibrium point**, a **critical point**, a **singular point**), shortly an **equilibrium** of Φ . A point that is not fixed is called **ordinary**.

Theorem 39 The following statements are equivalent for $\mu \in \mathbf{B}^n$:

$$\Phi(\mu) = \mu, \quad (3)$$

$$\exists \rho \in P_n, \forall t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu, \quad (4)$$

$$\forall \rho \in P_n, \forall t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu, \quad (5)$$

$$\exists \rho \in P_n, Or_\rho(\mu) = \{\mu\}, \quad (6)$$

$$\forall \rho \in P_n, Or_\rho(\mu) = \{\mu\}, \quad (7)$$

$$\mu \in NC_1 \cap \dots \cap NC_n. \quad (8)$$

Proof. (3) $\implies$ (4) We take $\rho \in P_n$ in the following way

$$\rho(t) = (1, \dots, 1) \cdot \chi_{\{t_0\}}(t) \oplus \dots \oplus (1, \dots, 1) \cdot \chi_{\{t_k\}}(t) \oplus \dots$$

with $(t_k) \in Seq$. For the sequence

$$\forall k \in \mathbf{N}, \alpha^k = (1, \dots, 1)$$

from Π_n we can prove by induction on k that

$$\forall k \in \mathbf{N}, \Phi^{\alpha^0 \dots \alpha^k}(\mu) = \mu \quad (9)$$

wherefrom

$$\Phi^\rho(\mu, t) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \mu \cdot \chi_{[t_0, t_1)}(t) \oplus \dots \oplus \mu \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots = \mu \quad (10)$$

(4) $\implies$ (3) From (4) we have the existence of $\alpha \in \Pi_n$ and $(t_k) \in Seq$ with the property that (10) is true, thus (9) is true. We denote

$$I_0 = \{i | i \in \{1, \dots, n\}, \alpha_i^0 = 1\},$$

$$I_1 = \{i | i \in \{1, \dots, n\}, \alpha_i^1 = 1\},$$

...

$$I_k = \{i | i \in \{1, \dots, n\}, \alpha_i^k = 1\},$$

...

and we have from (9):

$$\forall i \in \{1, \dots, n\},$$

$$\Phi_i^{\alpha^0}(\mu) = \begin{cases} \Phi_i(\mu), i \in I_0 \\ \mu_i, i \in \{1, \dots, n\} \setminus I_0 \end{cases} = \mu_i;$$

$$\forall i \in \{1, \dots, n\}, \Phi_i^{\alpha^0 \alpha^1}(\mu) = \Phi_i^{\alpha^1}(\Phi^{\alpha^0}(\mu)) =$$

$$= \Phi_i^{\alpha^1}(\mu) = \begin{cases} \Phi_i(\mu), i \in I_1 \\ \mu_i, i \in \{1, \dots, n\} \setminus I_1 \end{cases} = \mu_i;$$

...

$$\forall i \in \{1, \dots, n\}, \Phi_i^{\alpha^0 \alpha^1 \dots \alpha^k}(\mu) = \Phi_i^{\alpha^k}(\Phi^{\alpha^0 \dots \alpha^{k-1}}(\mu)) =$$

$$= \Phi_i^{\alpha^k}(\mu) = \begin{cases} \Phi_i(\mu), i \in I_k \\ \mu_i, i \in \{1, \dots, n\} \setminus I_k \end{cases} = \mu_i;$$

...

with the conclusion that

$$\forall k \in \mathbf{N}, \forall i \in I_0 \cup I_1 \cup \dots \cup I_k, \Phi_i(\mu) = \mu_i.$$

As α is progressive, some $k' \in \mathbf{N}$ exists with the property that

$$I_0 \cup I_1 \cup \dots \cup I_{k'} = \{1, \dots, n\},$$

thus (3) is true.

(3) $\implies$ (5) Let be

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus \dots \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus \dots \quad (11)$$

with $\alpha^0, \dots, \alpha^k, \dots \in \Pi_n$ and $(t_k) \in Seq$ arbitrary. It is proved by induction on k the validity of (9) and this implies the truth of (10).

(5) $\implies$ (3) This is true because (5) $\implies$ (4) and (4) $\implies$ (3) are true.

(4) $\iff$ (6) and (5) $\iff$ (7) are obvious.

(3) $\iff$ (8) $\Phi(\mu) = \mu \iff \Phi_1(\mu) = \mu_1$ and...and $\Phi_n(\mu) = \mu_n \iff \mu \in NC_1$ and...and $\mu \in NC_n \iff \mu \in NC_1 \cap \dots \cap NC_n$. ■

Definition 40 If $\Phi(\mu) = \mu$, then $\forall \rho \in P_n$, the orbit $\Phi^\rho(\mu, t) = \mu$ is called **rest position**.

8 Fixed points vs. final values of the orbits

Theorem 41 ([7], Theorem 49) The following fixed point property is true

$$\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n, \lim_{t \rightarrow \infty} \Phi^\rho(\mu, t) = \mu' \implies \Phi(\mu') = \mu'.$$

Proof. Let $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n, \rho \in P_n$ be arbitrary and fixed. Some $t' \in \mathbf{R}$ exists such that $\forall t \geq t'$,

$$\mu' = \Phi^\rho(\mu, t) \stackrel{\text{Remark 33}}{=} \Phi^{\rho \cdot \chi_{(t', \infty)}}(\mu', t).$$

Because $\forall t < t'$,

$$\Phi^{\rho \cdot \chi_{(t', \infty)}}(\mu', t) = \Phi^{(0, \dots, 0)}(\mu', t) = \mu',$$

from Theorem 39, (4) $\implies$ (3) we have that $\Phi(\mu') = \mu'$. ■

Remark 42 Theorem 41 shows that the final values of the states of the system Ξ_Φ are fixed points of Φ .

Theorem 43 ([7], Theorem 50) We have $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n$,

$$(\Phi(\mu') = \mu' \text{ and } \exists t' \in \mathbf{R}, \Phi^\rho(\mu, t') = \mu') \implies \forall t \geq t', \Phi^\rho(\mu, t) = \mu'.$$

Proof. For arbitrary $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n, \rho \in P_n$ we suppose that $\Phi(\mu') = \mu'$ and $\Phi^\rho(\mu, t') = \mu'$. We have $\forall t \geq t'$,

$$\Phi^\rho(\mu, t) \stackrel{\text{Remark 33}}{=} \Phi^{\rho \cdot \chi_{(t', \infty)}}(\mu', t) \stackrel{\text{Theorem 39, (3)} \implies (5)}{=} \mu'.$$

■

Remark 44 As resulting from Theorem 43, the accessible fixed points are final values of the states of the system Ξ_Φ .

The properties of the fixed points that are expressed by Theorems 39, 41, 43 give a better understanding of Example 35.

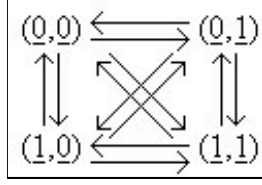


Figure 5: Transitivity

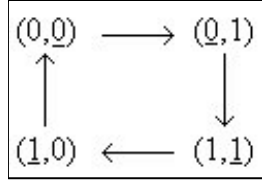


Figure 6: Transitivity

9 Transitivity

Definition 45 The system Ξ_Φ (or the function Φ) is **transitive**, or **minimal** if one of the following non-equivalent properties holds true:

$$\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \exists \rho \in P_n, \exists t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu', \quad (12)$$

$$\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n, \exists t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu'. \quad (13)$$

Remark 46 The property of transitivity may be considered one of surjectivity or one of accessibility.

If Φ is transitive, then it has no fixed points. For example $1_{\mathbf{B}^n}$ is not transitive since all $\mu \in \mathbf{B}^n$ are fixed points for this function.

Example 47 The property (12) of transitivity is exemplified in Figure 5 and the property (13) of transitivity is exemplified in Figure 6.

10 The equivalence of the systems

Notation 48 Let $h : \mathbf{B}^n \rightarrow \mathbf{B}^n$ and $x : \mathbf{R} \rightarrow \mathbf{B}^n$ be some functions. We denote by $h(x) : \mathbf{R} \rightarrow \mathbf{B}^n$ the function

$$\forall t \in \mathbf{R}, h(x)(t) = h(x(t)).$$

Remark 49 If $h : \mathbf{B}^n \rightarrow \mathbf{B}^n$ and $x \in S^{(n)}$ is expressed by

$$x(t) = x(-\infty+0) \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus \dots \oplus x(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots$$

then

$$\begin{aligned} h(x)(t) &= h(x(-\infty + 0)) \cdot \chi_{(-\infty, t_0)}(t) \oplus h(x(t_0)) \cdot \chi_{[t_0, t_1)}(t) \oplus \dots \\ &\dots \oplus h(x(t_k)) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots \end{aligned}$$

Notation 50 For $h : \mathbf{B}^n \rightarrow \mathbf{B}^n$ and $\alpha = \alpha^0, \dots, \alpha^k, \dots \in \mathbf{B}^n$, we denote by $\widehat{h}(\alpha)$ the sequence $h(\alpha^0), \dots, h(\alpha^k), \dots \in \mathbf{B}^n$.

Notation 51 Let be $k \geq 2$ arbitrary and we denote for $\mu^1, \dots, \mu^k \in \mathbf{B}^n$,

$$\mu^1 \cup \dots \cup \mu^k = (\mu_1^1 \cup \dots \cup \mu_1^k, \dots, \mu_n^1 \cup \dots \cup \mu_n^k).$$

Notation 52 We denote by Ω_n the set of the functions $h : \mathbf{B}^n \rightarrow \mathbf{B}^n$ that fulfill

- i) h is bijective;
- ii) $h(0, \dots, 0) = (0, \dots, 0)$, $h(1, \dots, 1) = (1, \dots, 1)$;
- iii) $\forall k \geq 2, \forall \mu^1 \in \mathbf{B}^n, \dots, \forall \mu^k \in \mathbf{B}^n$,

$$\mu^1 \cup \dots \cup \mu^k = (1, \dots, 1) \iff h(\mu^1) \cup \dots \cup h(\mu^k) = (1, \dots, 1).$$

Theorem 53 a) Ω_n is group relative to the composition ' $\circ$ ' of the functions;

- b) $\forall h \in \Omega_n, \forall \alpha \in \Pi_n, \widehat{h}(\alpha) \in \Pi_n$;
- c) $\forall h \in \Omega_n, \forall \rho \in P_n, h(\rho) \in P_n$.

Proof. a) We can prove the fact that $1_{\mathbf{B}^n} \in \Omega_n$, $\forall h \in \Omega_n, \forall h' \in \Omega_n, h \circ h' \in \Omega_n$ and $\forall h \in \Omega_n, h^{-1} \in \Omega_n$. For example let be $h \in \Omega_n, k \geq 2$ and $\nu^1, \dots, \nu^k \in \mathbf{B}^n$ arbitrary, for which we denote $\mu^1 = h^{-1}(\nu^1), \dots, \mu^k = h^{-1}(\nu^k)$. We have:

$$\begin{aligned} h^{-1}(\nu^1 \cup \dots \cup \nu^k) &= (1, \dots, 1) \iff \nu^1 \cup \dots \cup \nu^k = h(1, \dots, 1) = (1, \dots, 1) \\ &\iff h(\mu^1) \cup \dots \cup h(\mu^k) = (1, \dots, 1) \iff \mu^1 \cup \dots \cup \mu^k = (1, \dots, 1) \\ &\iff h^{-1}(\nu^1) \cup \dots \cup h^{-1}(\nu^k) = (1, \dots, 1), \end{aligned}$$

thus h^{-1} fulfills iii) from Notation 52.

b) Let $h \in \Omega_n$ and $\alpha = \alpha^0, \dots, \alpha^k, \dots \in \mathbf{B}^n$ be arbitrary. We denote for $p \geq 1$

$$\{\mu^1, \dots, \mu^p\} = \{\mu \mid \mu \in \mathbf{B}^n, \{k \mid k \in \mathbf{N}, \alpha^k = \mu\} \text{ is infinite}\}$$

and we remark that

$$\alpha \in \Pi_n \iff \mu^1, \dots, \mu^p, \mu^1, \dots, \mu^p, \mu^1, \dots \in \Pi_n \iff$$

$$\iff \begin{cases} \mu^1 = (1, \dots, 1), p = 1 \\ \mu^1 \cup \dots \cup \mu^p = (1, \dots, 1), p \geq 2 \end{cases},$$

$$\begin{aligned} \widehat{h}(\alpha) \in \Pi_n &\iff h(\mu^1), \dots, h(\mu^p), h(\mu^1), \dots, h(\mu^p), h(\mu^1), \dots \in \Pi_n \iff \\ &\iff \begin{cases} h(\mu^1) = (1, \dots, 1), p = 1 \\ h(\mu^1) \cup \dots \cup h(\mu^p) = (1, \dots, 1), p \geq 2 \end{cases}. \end{aligned}$$

Case $p = 1$,

$$\alpha \in \Pi_n \implies \mu^1 = (1, \dots, 1) \implies h(\mu^1) = (1, \dots, 1) \implies \widehat{h}(\alpha) \in \Pi_n.$$

Case $p \geq 2$,

$$\begin{aligned} \alpha \in \Pi_n \implies \mu^1 \cup \dots \cup \mu^p = (1, \dots, 1) \implies h(\mu^1) \cup \dots \cup h(\mu^p) = (1, \dots, 1) \implies \\ \implies \widehat{h}(\alpha) \in \Pi_n. \end{aligned}$$

c) Let us take arbitrarily some $h \in \Omega_n$ and a function $\rho \in P_n$,

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus \dots \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus \dots$$

where $\alpha \in \Pi_n$ and $(t_k) \in Seq$. We have

$$\begin{aligned} h(\rho)(t) &= h(\rho(t)) = \\ &= h((0, \dots, 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus (0, \dots, 0) \cdot \chi_{(t_0, t_1)}(t) \oplus \dots \\ &\quad \dots \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus (0, \dots, 0) \cdot \chi_{(t_k, t_{k+1})}(t) \oplus \dots) \\ &= h(0, \dots, 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus h(\alpha^0) \cdot \chi_{\{t_0\}}(t) \oplus h(0, \dots, 0) \cdot \chi_{(t_0, t_1)}(t) \oplus \dots \\ &\quad \dots \oplus h(\alpha^k) \cdot \chi_{\{t_k\}}(t) \oplus h(0, \dots, 0) \cdot \chi_{(t_k, t_{k+1})}(t) \oplus \dots \\ &= h(\alpha^0) \cdot \chi_{\{t_0\}}(t) \oplus \dots \oplus h(\alpha^k) \cdot \chi_{\{t_k\}}(t) \oplus \dots \end{aligned}$$

Because $\widehat{h}(\alpha) \in \Pi_n$, taking into account b), we conclude that $h(\rho) \in P_n$.

■

Theorem 54 *Let be the functions $\Phi, \Psi : \mathbf{B}^n \rightarrow \mathbf{B}^n$ and the bijections $h : \mathbf{B}^n \rightarrow \mathbf{B}^n, h' \in \Omega_n$. The following statements are equivalent:*

a) $\forall \nu \in \mathbf{B}^n$, the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \xrightarrow{\Phi^\nu} & \mathbf{B}^n \\ h \downarrow & & \downarrow h \\ \mathbf{B}^n & \xrightarrow{\Psi^{h'(\nu)}} & \mathbf{B}^n \end{array}$$

is commutative;

b) $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \Pi_n, \forall k \in \mathbf{N}_+$,

$$h(\widehat{\Phi}^\alpha(\mu, k)) = \widehat{\Psi}^{\widehat{h}'(\alpha)}(h(\mu), k);$$

c) $\forall \mu \in \mathbf{B}^n, \forall \rho \in P_n, \forall t \in \mathbf{R}$,

$$h(\Phi^\rho(\mu, t)) = \Psi^{h'(\rho)}(h(\mu), t). \quad (14)$$

Proof. a) $\implies$ b) It is sufficient to prove that $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \Pi_n, \forall k \in \mathbf{N}$,

$$h(\Phi^{\alpha^0 \dots \alpha^k}(\mu)) = \Psi^{h'(\alpha^0) \dots h'(\alpha^k)}(h(\mu)) \quad (15)$$

since this is equivalent with b).

We fix arbitrarily some μ and some α and we use the induction on k . For $k = 0$ the statement is proved, thus we suppose that it is true for k and we prove it for $k + 1$:

$$\begin{aligned} h(\Phi^{\alpha^0 \dots \alpha^k \alpha^{k+1}}(\mu)) &= h(\Phi^{\alpha^{k+1}}(\Phi^{\alpha^0 \dots \alpha^k}(\mu))) = \Psi^{h'(\alpha^{k+1})}(h(\Phi^{\alpha^0 \dots \alpha^k}(\mu))) = \\ &= \Psi^{h'(\alpha^{k+1})}(\Psi^{h'(\alpha^0) \dots h'(\alpha^k)}(h(\mu))) = \Psi^{h'(\alpha^0) \dots h'(\alpha^k) h'(\alpha^{k+1})}(h(\mu)). \end{aligned}$$

b) $\implies$ c) For arbitrary $\mu \in \mathbf{B}^n$ and $\rho \in P_n$,

$$\rho(t) = \rho(t_0) \cdot \chi_{\{t_0\}}(t) \oplus \dots \oplus \rho(t_k) \cdot \chi_{\{t_k\}}(t) \oplus \dots$$

with $(t_k) \in Seq, \rho(t_0), \dots, \rho(t_k), \dots \in \Pi_n$, we have that

$$h'(\rho)(t) = h'(\rho(t)) = h'(\rho(t_0)) \cdot \chi_{\{t_0\}}(t) \oplus \dots \oplus h'(\rho(t_k)) \cdot \chi_{\{t_k\}}(t) \oplus \dots \quad (16)$$

is an element of P_n (see Theorem 53 c)) and

$$\begin{aligned} h(\Phi^\rho(\mu, t)) &= h(\mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \Phi^{\rho(t_0)}(\mu) \cdot \chi_{[t_0, t_1)}(t) \oplus \dots \\ &\quad \dots \oplus \Phi^{\rho(t_0) \dots \rho(t_k)}(\mu) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots) = \\ &= h(\mu) \cdot \chi_{(-\infty, t_0)}(t) \oplus h(\Phi^{\rho(t_0)}(\mu)) \cdot \chi_{[t_0, t_1)}(t) \oplus \dots \\ &\quad \dots \oplus h(\Phi^{\rho(t_0) \dots \rho(t_k)}(\mu)) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots = \\ &\stackrel{(15)}{=} h(\mu) \cdot \chi_{(-\infty, t_0)}(t) \oplus \Psi^{h'(\rho(t_0))}(h(\mu)) \cdot \chi_{[t_0, t_1)}(t) \oplus \dots \\ &\quad \dots \oplus \Psi^{h'(\rho(t_0)) \dots h'(\rho(t_k))}(h(\mu)) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots \stackrel{(16)}{=} \Psi^{h'(\rho)}(h(\mu), t). \end{aligned}$$

c) $\implies$ a) Let $\nu, \mu \in \mathbf{B}^n$ be arbitrary and fixed and we consider $\rho \in P_n$,

$$\rho(t) = \nu \cdot \chi_{\{t_0\}}(t) \oplus \rho(t_1) \cdot \chi_{\{t_1\}}(t) \oplus \dots \oplus \rho(t_k) \cdot \chi_{\{t_k\}}(t) \oplus \dots$$

with $(t_k) \in Seq$ fixed too. We have

$$\begin{aligned} h(\Phi^\rho(\mu, t)) &= h(\mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \Phi^\nu(\mu) \cdot \chi_{[t_0, t_1)}(t) \oplus \Phi^{\nu \rho(t_1)}(\mu) \cdot \chi_{[t_1, t_2)}(t) \oplus \dots) = \\ &= h(\mu) \cdot \chi_{(-\infty, t_0)}(t) \oplus h(\Phi^\nu(\mu)) \cdot \chi_{[t_0, t_1)}(t) \oplus h(\Phi^{\nu \rho(t_1)}(\mu)) \cdot \chi_{[t_1, t_2)}(t) \oplus \dots \end{aligned}$$

But

$$h'(\rho)(t) = h'(\rho(t)) = h'(\nu) \cdot \chi_{\{t_0\}}(t) \oplus h'(\rho(t_1)) \cdot \chi_{\{t_1\}}(t) \oplus \dots,$$

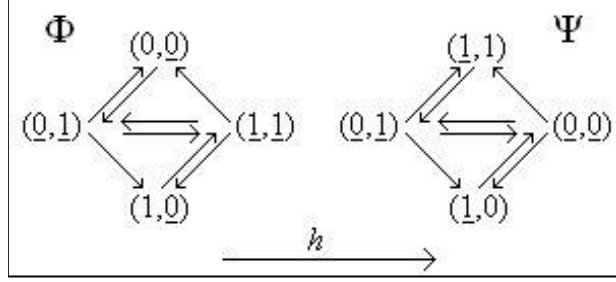


Figure 7: Equivalent systems

$$\begin{aligned} \Psi^{h'(\rho)}(h(\mu), t) &= \\ &= h(\mu) \cdot \chi_{(-\infty, t_0)}(t) \oplus \Psi^{h'(\nu)} \cdot \chi_{[t_0, t_1)}(t) \oplus \Psi^{h'(\nu)h'(\rho(t_1))} \cdot \chi_{[t_1, t_2)}(t) \oplus \dots \end{aligned}$$

and from (14), for $t \in [t_0, t_1)$, we obtain

$$h(\Phi^\nu(\mu)) = \Psi^{h'(\nu)}(h(\mu)).$$

■

Definition 55 We consider the functions $\Phi, \Psi : \mathbf{B}^n \rightarrow \mathbf{B}^n$. If two bijections $h : \mathbf{B}^n \rightarrow \mathbf{B}^n, h' \in \Omega_n$ exist such that one of the equivalent properties a), b), c) from Theorem 54 is satisfied, then Ξ_Φ, Ξ_Ψ are called **equivalent** and Φ, Ψ are called **conjugated**. In this case we denote $\Phi \xrightarrow{(h, h')} \Psi$.

Remark 56 The equivalence of the universal regular autonomous asynchronous systems is indeed an equivalence and it should be understood as a change of coordinates. Thus Φ and Ψ are indistinguishable.

Example 57 $\Phi, \Psi : \mathbf{B}^2 \rightarrow \mathbf{B}^2$ are given by, see Figure 7

$$\forall(\mu_1, \mu_2) \in \mathbf{B}^2, \Phi(\mu_1, \mu_2) = (\mu_1 \oplus \mu_2, \overline{\mu_2}),$$

$$\forall(\mu_1, \mu_2) \in \mathbf{B}^2, \Psi(\mu_1, \mu_2) = (\overline{\mu_1}, \overline{\mu_1} \cdot \overline{\mu_2} \cup \mu_1 \cdot \mu_2)$$

and the bijection $h : \mathbf{B}^2 \rightarrow \mathbf{B}^2$ is

$$\forall(\mu_1, \mu_2) \in \mathbf{B}^2, h(\mu_1, \mu_2) = (\overline{\mu_2}, \overline{\mu_1}).$$

The diagram

$$\begin{array}{ccc} \mathbf{B}^2 & \xrightarrow{\Phi^\nu} & \mathbf{B}^2 \\ h \downarrow & & \downarrow h \\ \mathbf{B}^2 & \xrightarrow{\Psi^{\nu'}} & \mathbf{B}^2 \end{array}$$

commutes for $\nu = \nu' = (0, 0)$ and on the other hand for $\nu = \nu' = (1, 1)$ we have the assignments

$$\begin{array}{cccc} (0, 0) \xrightarrow{\Phi} (0, 1) & (0, 1) \xrightarrow{\Phi} (1, 0) & (1, 0) \xrightarrow{\Phi} (1, 1) & (1, 1) \xrightarrow{\Phi} (0, 0) \\ h \downarrow & \downarrow h, & h \downarrow & \downarrow h, & h \downarrow & \downarrow h, & h \downarrow & \downarrow h. \\ (1, 1) \xrightarrow{\Psi} (0, 1) & (0, 1) \xrightarrow{\Psi} (1, 0) & (1, 0) \xrightarrow{\Psi} (0, 0) & (0, 0) \xrightarrow{\Psi} (1, 1) \end{array}$$

We denote $\pi_i : \mathbf{B}^2 \rightarrow \mathbf{B}, \forall (\mu_1, \mu_2) \in \mathbf{B}^2$,

$$\pi_i(\mu_1, \mu_2) = \mu_i, i = \overline{1, 2}.$$

For $\nu = (0, 1), \nu' = (1, 0)$ we have

$$\begin{array}{cccc} (0, 0) \xrightarrow{(\pi_1, \Phi_2)} (0, 1) & (0, 1) \xrightarrow{(\pi_1, \Phi_2)} (0, 0) & (1, 0) \xrightarrow{(\pi_1, \Phi_2)} (1, 1) & (1, 1) \xrightarrow{(\pi_1, \Phi_2)} (1, 0) \\ h \downarrow & \downarrow h, & h \downarrow & \downarrow h, & h \downarrow & \downarrow h, & h \downarrow & \downarrow h \\ (1, 1) \xrightarrow{(\Psi_1, \pi_2)} (0, 1) & (0, 1) \xrightarrow{(\Psi_1, \pi_2)} (1, 1) & (1, 0) \xrightarrow{(\Psi_1, \pi_2)} (0, 0) & (0, 0) \xrightarrow{(\Psi_1, \pi_2)} (1, 0) \end{array}$$

and for $\nu = (1, 0), \nu' = (0, 1)$ the assignments are

$$\begin{array}{cccc} (0, 0) \xrightarrow{(\Phi_1, \pi_2)} (0, 0) & (0, 1) \xrightarrow{(\Phi_1, \pi_2)} (1, 1) & (1, 0) \xrightarrow{(\Phi_1, \pi_2)} (1, 0) & (1, 1) \xrightarrow{(\Phi_1, \pi_2)} (0, 1) \\ h \downarrow & \downarrow h, & h \downarrow & \downarrow h, & h \downarrow & \downarrow h, & h \downarrow & \downarrow h \\ (1, 1) \xrightarrow{(\pi_1, \Psi_2)} (1, 1) & (0, 1) \xrightarrow{(\pi_1, \Psi_2)} (0, 0) & (1, 0) \xrightarrow{(\pi_1, \Psi_2)} (1, 0) & (0, 0) \xrightarrow{(\pi_1, \Psi_2)} (0, 1) \end{array}$$

respectively. Φ and Ψ are conjugated.

Example 58 The functions $h, h' : \mathbf{B}^2 \rightarrow \mathbf{B}^2$ are given in the following table

(μ_1, μ_2)	$h(\mu_1, \mu_2)$	$h'(\mu_1, \mu_2)$
(0, 0)	(0, 1)	(0, 0)
(0, 1)	(1, 1)	(1, 0)
(1, 0)	(0, 0)	(0, 1)
(1, 1)	(1, 0)	(1, 1)

and the state portraits of the two systems are given in Figure 8. Ξ_Φ and Ξ_Ψ are equivalent.

Theorem 59 If Φ and Ψ are conjugated, then the following possibilities exist:

- a) $\Phi = \Psi = 1_{\mathbf{B}^n}$;
- b) $\Phi \neq 1_{\mathbf{B}^n}$ and $\Psi \neq 1_{\mathbf{B}^n}$.

Proof. We presume that $\Phi \xrightarrow{(h, h')} \Psi$. In the equation

$$\forall \nu \in \mathbf{B}^n, \forall \mu \in \mathbf{B}^n, h(\Phi^\nu(\mu)) = \Psi^{h'(\nu)}(h(\mu))$$

we put $\Psi = 1_{\mathbf{B}^n}$ and we have

$$\forall \nu \in \mathbf{B}^n, \forall \mu \in \mathbf{B}^n, h(\Phi^\nu(\mu)) = h(\mu)$$

thus $\forall \nu \in \mathbf{B}^n, \Phi^\nu = 1_{\mathbf{B}^n}$ and finally $\Phi = 1_{\mathbf{B}^n}$. ■

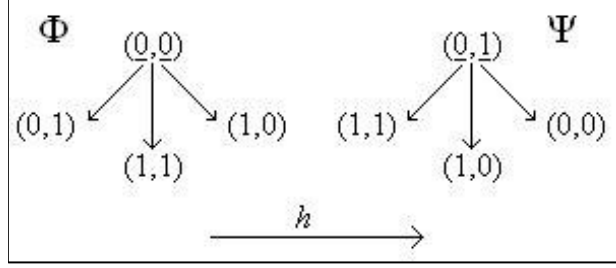


Figure 8: Equivalent systems

Theorem 60 *We suppose that Ξ_Φ and Ξ_Ψ are equivalent and let be h, h' such that $\Phi \xrightarrow{(h, h')} \Psi$.*

- a) *If μ is a fixed point of Φ , then $h(\mu)$ is a fixed point of Ψ .*
- b) *For any $\mu \in \mathbf{B}^n$ and any $\rho \in P_n$, if $\Phi^\rho(\mu, t)$ is periodical with the period T_0 , then $\Psi^{h'(\rho)}(h(\mu), t)$ is periodical with the period T_0 .*
- c) *If Ξ_Φ is transitive, then Ξ_Ψ is transitive.*

Proof. a) We suppose that $\Phi(\mu) = \mu$. The commutativity of the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \xrightarrow{\Phi^\nu} & \mathbf{B}^n \\ h \downarrow & & \downarrow h \\ \mathbf{B}^n & \xrightarrow{\Psi^{h'(\nu)}} & \mathbf{B}^n \end{array}$$

for $\nu = (1, \dots, 1)$ gives

$$\begin{aligned} h(\mu) &= h(\Phi(\mu)) = h(\Phi^{(1, \dots, 1)}(\mu)) = \Psi^{h'(1, \dots, 1)}(h(\mu)) = \\ &= \Psi^{(1, \dots, 1)}(h(\mu)) = \Psi(h(\mu)). \end{aligned}$$

b) Let be $\mu \in \mathbf{B}^n$ and $\rho \in P_n$. The hypothesis states that $\exists t' \in \mathbf{R}, \forall t \geq t',$

$$\Phi^\rho(\mu, t) = \Phi^\rho(\mu, t + T_0)$$

and in this situation

$$\Psi^{h'(\rho)}(h(\mu), t) = h(\Phi^\rho(\mu, t)) = h(\Phi^\rho(\mu, t + T_0)) = \Psi^{h'(\rho)}(h(\mu), t + T_0).$$

c) Let $\mu, \mu' \in \mathbf{B}^n$ be arbitrary and fixed. The hypothesis (12) states that

$$\exists \rho \in P_n, \exists t \in \mathbf{R}, \Phi^\rho(h^{-1}(\mu), t) = h^{-1}(\mu'),$$

wherefrom

$$\Psi^{h'(\rho)}(\mu, t) = \Psi^{h'(\rho)}(h(h^{-1}(\mu)), t) = h(\Phi^\rho(h^{-1}(\mu), t) = h(h^{-1}(\mu')) = \mu'.$$

The situation with (13) is similar. ■

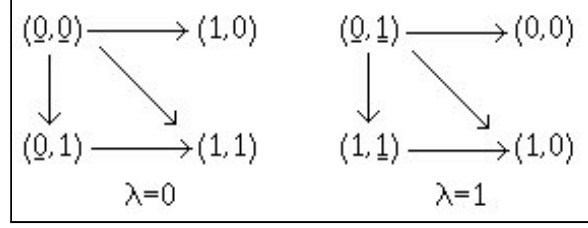


Figure 9: Structural stability

11 Dynamical bifurcations

Definition 61 We consider the case when the generator function $\Phi : \mathbf{B}^n \times \mathbf{B}^m \rightarrow \mathbf{B}^n$, $\mathbf{B}^n \times \mathbf{B}^m \ni (\mu, \lambda) \rightarrow \Phi(\mu, \lambda) \in \mathbf{B}^n$ of $\Xi_{\Phi(\cdot, \lambda)}$ depends on the parameter $\lambda \in \mathbf{B}^m$. We fix λ and let be $\lambda' \in \mathbf{B}^m$. If $\Phi(\cdot, \lambda)$ and $\Phi(\cdot, \lambda')$ are conjugated, then $\Phi(\cdot, \lambda')$ is called an **admissible** (or **allowable**) **perturbation** of $\Phi(\cdot, \lambda)$.

Remark 62 Intuitively speaking (Ott, [2]) a dynamical bifurcation is a qualitative change in the dynamic of the system $\Xi_{\Phi(\cdot, \lambda)}$ that occurs at the variation of the parameter λ .

Definition 63 If for any parameters $\lambda, \lambda' \in \mathbf{B}^m$ the systems $\Xi_{\Phi(\cdot, \lambda)}$ and $\Xi_{\Phi(\cdot, \lambda')}$ are equivalent, then Φ is called **structurally stable**; the existence of λ, λ' such that $\Xi_{\Phi(\cdot, \lambda)}$ and $\Xi_{\Phi(\cdot, \lambda')}$ are not equivalent is called a **dynamical bifurcation**.

Equivalently, let us fix an arbitrary $\lambda \in \mathbf{B}^m$. If $\forall \lambda' \in \mathbf{B}^m$, $\Phi(\cdot, \lambda')$ is an admissible perturbation of $\Phi(\cdot, \lambda)$, then Φ is said to be **structurally stable**, otherwise we say that Φ has a **dynamical bifurcation**.

Remark 64 If $\forall \lambda \in \mathbf{B}^m, \forall \lambda' \in \mathbf{B}^m$ the bijections $h : \mathbf{B}^n \rightarrow \mathbf{B}^n, h' \in \Omega_n$ exist such that $\forall \nu \in \mathbf{B}^n$, the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \xrightarrow{\Phi^\nu(\cdot, \lambda)} & \mathbf{B}^n \\ h \downarrow & & \downarrow h \\ \mathbf{B}^n & \xrightarrow{\Phi^{h'(\nu)}(\cdot, \lambda')} & \mathbf{B}^n \end{array}$$

commutes, then Φ is structurally stable, otherwise we have a dynamical bifurcation.

Example 65 In Figure 9 ($n = 2, m = 1$), Φ is structurally stable and

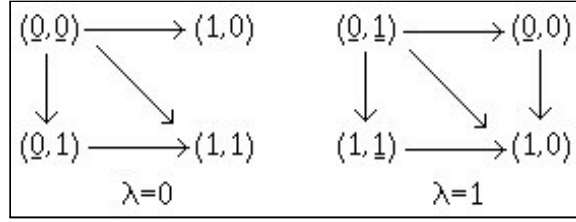


Figure 10: Dynamical bifurcation

the bijections h, h' are defined accordingly to the following table:

(μ_1, μ_2)	$h(\mu_1, \mu_2)$	$h'(\mu_1, \mu_2)$
$(0, 0)$	$(0, 1)$	$(0, 0)$
$(0, 1)$	$(1, 1)$	$(1, 0)$
$(1, 0)$	$(0, 0)$	$(0, 1)$
$(1, 1)$	$(1, 0)$	$(1, 1)$

Example 66 In Figure 10 ($n = 2, m = 1$), Φ has a dynamical bifurcation.

Definition 67 The **bifurcation diagram** is a partition of the set of systems $\{\Xi_{\Phi(\cdot, \lambda)} | \lambda \in \mathbf{B}^m\}$ in classes of equivalence given by the equivalence of the systems, together with representative state portraits for each class of equivalence.

Example 68 Figure 10 is a bifurcation diagram.

Definition 69 The **bifurcation diagram** ([2], page 5) is the graph that gives the position of the fixed points depending on a parameter, such that a bifurcation exists.

Remark 70 Such a(n informal) definition works for calling Figure 10 a bifurcation diagram, since there fixed points exist. However for Figure 11 this definition does not work, because a bifurcation exists there, but no fixed points.

Definition 71 Let be $\Phi, \Psi : \mathbf{B}^n \times \mathbf{B}^m \rightarrow \mathbf{B}^n$. The families of systems $(\Xi_{\Phi(\cdot, \lambda)})_{\lambda \in \mathbf{B}^m}$ and $(\Xi_{\Psi(\cdot, \lambda)})_{\lambda \in \mathbf{B}^m}$ are called **equivalent** if there exists a bijection $h'' : \mathbf{B}^m \rightarrow \mathbf{B}^m$ such that $\forall \lambda \in \mathbf{B}^m, \Xi_{\Phi(\cdot, \lambda)}$ and $\Xi_{\Psi(\cdot, h''(\lambda))}$ are equivalent in the sense of Definition 55.

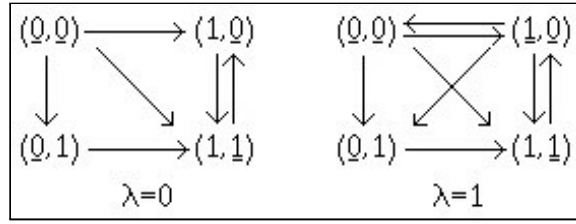


Figure 11: Dynamical bifurcation

References

- [1] Constanta-Dana Constantinescu, Chaos, fractals and applications, the Flower Power publishing house, Pitesti, 2003 (in Romanian).
- [2] Marius-Florin Danca, Logistic map: dynamics, bifurcation and chaos, the publishing house of the Pitesti University, Pitesti, 2001 (in Romanian).
- [3] Adelina Georgescu, Mihnea Moroianu, Iuliana Oprea, Bifurcation theory, principles and applications, the publishing house of the Pitesti University, Pitesti, 1999 (in Romanian).
- [4] Yuri A. Kuznetsov, Elements of Applied Bifurcation Theory, Second Edition, Springer, 1997.
- [5] Mihaela Sterpu, Dynamic and bifurcation for two generalized van der Pol models, the publishing house of the Pitesti University, Pitesti, 2001 (in Romanian).
- [6] Mariana P. Trifan, Dynamic and bifurcation in the mathematical study of the cancer, the Pamantul publishing house, Pitesti, 2006 (in Romanian).
- [7] Serban E. Vlad, Boolean dynamical systems, Romai Journal, Vol. 3, Nr. 2, 2007.