Boolean dynamical systems

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Abstract

The asynchronous circuits from the digital electrical engineering are modeled by the so-called asynchronous systems. An autonomous asynchronous system consists in a set X of 'nice' $\mathbf{R} \to \{0,1\}^n$ functions $n \geq 1$, called signals, representing non-deterministically the models of the tensions that describe the behavior of an asynchronous circuit without inputs. A special case of such a system X is the one when a function $\Phi : \{0,1\}^n \to \{0,1\}^n$ is given such that any $x \in X$ fulfills a 'differential' equation involving Φ. In such conditions, by analogy with the 'real' semi-dynamical systems, the Boolean dynamical systems may be defined.

Our paper defines the nullclins, the motions and their speed, the invariant subsets of $\{0,1\}^n$, the attraction, the limit cycles, the evolutions of $x \in X$, the stable manifolds, the Huffman systems and several properties of invariance which are very important in electrical engineering (such as the delay insensitivity). The relation with the discrete time systems is also suggested.

1 Preliminaries

Notation 1 *Let be the arbitrary set M. The following notation will be useful:*

$$P^*(M) = \{M' | M' \subset M, M' \neq \emptyset\}.$$

Notation 2 *The set of the subsequences of the natural numbers set $\mathbf{N}$ is denoted by $Sub(\mathbf{N})$:*

$$Sub(\mathbf{N}) = \{(j_k) | j_k \in \mathbf{N}, k \in \mathbf{N}, j_0 < ... < j_k < ...\}.$$

Definition 3 *The set* $\mathbf{B}=\{0,1\}$ *is endowed with the order* $0\leq 1$ *and with the usual laws* $^-,\cdot,\cup,\oplus$*. It is called the* ***binary Boole algebra****.*

Remark 4 *The set* $\mathbf{B}$ *is a Boole algebra indeed relative to* $^-,\cdot,\cup$ *and it is also a field relative to* $\oplus,\cdot$. $\mathbf{B}^n$*, together with the sum* $\oplus$ *made coordinatewise and with the scalar product* $\cdot$*, is a linear space over* $\mathbf{B}$*.*

Notation 5 *We denote by*

$$\varepsilon^i=(0,...,\underset{i}{1},...,0),i=\overline{1,n}$$

the vectors of the canonical base of $\mathbf{B}^n$*.*

Notation 6 *The modulo 2 summation of the vectors* $v^j\in\mathbf{B}^n,j\in J$ *(*J *must be a finite set) is denoted by* $\underset{j\in J}{\Xi}v^j$*. By definition* $\underset{j\in\emptyset}{\Xi}v^j=0$*.*

Definition 7 *If* $\alpha:\mathbf{N}\rightarrow\mathbf{B}^n$ *is a sequence,* $\alpha(k)\overset{not}{=}\alpha^k,k\in\mathbf{N}$*, then its* ***limit*** $\lim\limits_{k\rightarrow\infty}\alpha^k\in\mathbf{B}^n$ *is defined by the property*

$$\exists k'\in\mathbf{N},\forall k''\geq k',\alpha^{k''}=\lim_{k\rightarrow\infty}\alpha^k.$$

Definition 8 *Let be the function* $x:\mathbf{R}\rightarrow\mathbf{B}^n$*. Its* ***initial value*** $\lim\limits_{t\rightarrow-\infty}x(t)\in\mathbf{B}^n$ *(also denoted by* $x(-\infty+0)$*) and its* ***final value*** $\lim\limits_{t\rightarrow\infty}x(t)\in\mathbf{B}^n$ *(denoted sometimes by* $x(\infty-0)$*) are defined by*

$$\exists t'\in\mathbf{R},\forall t''\leq t',x(t'')=\lim_{t\rightarrow-\infty}x(t),$$

$$\exists t'\in\mathbf{R},\forall t''\geq t',x(t'')=\lim_{t\rightarrow\infty}x(t).$$

Notation 9 *We denote by* $\tau^d:\mathbf{R}\rightarrow\mathbf{R}$*,* $d\in\mathbf{R}$ *the* ***translation***

$$\forall t\in\mathbf{R},\tau^d(t)=t-d.$$

Thus for any $x:\mathbf{R}\rightarrow\mathbf{B}^n$*, we denote by* $x\circ\tau^d:\mathbf{R}\rightarrow\mathbf{B}^n$ *the function*

$$\forall t\in\mathbf{R},(x\circ\tau^d)(t)=x(t-d).$$

Definition 10 *The* ***characteristic function*** $\chi_A:\mathbf{R}\rightarrow\mathbf{B}$ *of the set* $A\subset\mathbf{R}$ *is given by*

$$\forall t\in\mathbf{R},\chi_A(t)=\begin{cases}1,t\in A\\0,\ else\end{cases}.$$

Notation 11 *We use the following notation*

$Seq = \{(t_k) | t_k \in \mathbf{R}, k \in \mathbf{N},\ t_0 < ... < t_k < ...$ *is unbounded from above*$\}$.

Definition 12 *The* ***cyclic values*** $\nu \in \mathbf{B}^n$ *and the* ***co-cyclic values*** $\mu \in \mathbf{B}^n$ *of* $x : \mathbf{R} \to \mathbf{B}^n$ *are defined by*

$$\exists (t_k) \in Seq, \forall k \in \mathbf{N}, x(t_k) = \nu,$$

$$\exists (t_k) \in Seq, \forall k \in \mathbf{N}, x(-t_k) = \mu.$$

Remark 13 *The initial and the final values of* $x : \mathbf{R} \to \mathbf{B}^n$ *may not exist and if any of them exists, then it is unique. The cyclic and the co-cyclic values of* x *always exist and they are not unique in general. If* x *has a unique cyclic (co-cyclic) value* ν (μ)*, then* $\nu = x(\infty - 0)$ $(\mu = x(-\infty + 0))$*.*

Definition 14 *A function* $x : \mathbf{R} \to \mathbf{B}^n$ *is called* $n-$***signal****, shortly* ***signal*** *if* $\mu \in B^n$ *and* $(t_k) \in Seq$ *exist so that*

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... \quad (1)$$

where in (1) we have abusively used the same symbols $\cdot, \oplus$ *for the laws that are induced by those of* $\mathbf{B}$*. The set of the* $n-$*signals is denoted by* $S^{(n)}$ *and instead of* $S^{(1)}$ *we usually write* S*.*

Definition 15 *Let* $x \in S^{(n)}$ *be given by (1). Its* ***left limit*** $x(t-0)$ *is the* $\mathbf{R} \to \mathbf{B}^n$ *function defined as*

$$x(t-0) = \mu \cdot \chi_{(-\infty,t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0,t_1]}(t) \oplus ... \oplus x(t_k) \cdot \chi_{(t_k,t_{k+1}]}(t) \oplus ... \quad (2)$$

Definition 16 *By definition, the* ***left derivative*** *of* $x \in S^{(n)}$ *is the function* $Dx : \mathbf{R} \to \mathbf{B}^n$,

$$Dx(t) = x(t-0) \oplus x(t).$$

Remark 17 *From (1) and (2) we infer*

$$Dx(t) = (\mu \oplus x(t_0)) \cdot \chi_{\{t_0\}}(t) \oplus (x(t_0) \oplus x(t_1)) \cdot \chi_{\{t_1\}}(t) \oplus ...$$

$$... \oplus (x(t_{k-1}) \oplus x(t_k)) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

On the other hand the right limit and the right derivative of $x \in S^{(n)}$*:*

$$x(t+0) = x(t),$$

$$D^* x(t) = x(t+0) \oplus x(t) = 0$$

will not be important in our work.

Definition 18 *Let be* $U \in P^*(S^{(m)})$. *A multi-valued function* $f : U \rightarrow P^*(S^{(n)})$ *is called* ***asynchronous system****, shortly* ***system****. Any* $u \in U$ *is called (****admissible****)* ***input*** *and the functions* $x \in f(u)$ *are called (****possible****)* ***states****.*

Definition 19 *Any of*

a) a system f *with* $\exists X \in P^*(S^{(n)}), \forall u \in U, f(u) = X$,

b) a system f *having the property that* U *has a single element,*

c) a set $X \in P^*(S^{(n)})$

is called ***autonomous (asynchronous) system****.*

Remark 20 *The concept of system originates in the modeling of the asynchronous circuits. The multi-valued character of the cause-effect association is due to the statistical fluctuations in the fabrication process, the variations in the ambiental temperature, the power supply etc. Sometimes the systems are given by equations and/or inequalities.*

An autonomous system is interpreted as a system without input and we prefer to use for this the concept from Definition 19 c).

2 Semi-dynamical systems. An example

Definition 21 *Let* M *be a set. A family of applications* $\Upsilon_t : M \rightarrow M, t \in [0, \infty)$ *that fulfills the conditions*

$$\Upsilon_0 = 1_M, \tag{3}$$

$$\forall t \in [0, \infty), \forall t' \in [0, \infty), \Upsilon_{t+t'} = \Upsilon_t \circ \Upsilon_{t'} \tag{4}$$

is called ***semi-group of maps*** *of* M *with one parameter.*

Definition 22 *A couple* $\gamma = (M, \Upsilon = (\Upsilon_t)_{t \in [0,\infty)})$*, where* M *is some set and* $\Upsilon : [0, \infty) \rightarrow M^M, \Upsilon(t) \overset{not}{=} \Upsilon_t, t \in [0, \infty)$ *defines a semi-group of maps of* M *with one parameter, is called a* ***semi-dynamical system*** *and the parameter* t *is called* ***time****. The set* M *is called the* ***phase space*** *or the* ***state space*** *and the points* $x \in M$ *are called* ***phases*** *or* ***states****.*

Remark 23 *A process is called deterministic in [1] if its whole future evolution is uniquely determined by the present evolution. The semi-dynamical systems are mathematical models of the deterministic processes.*

Our purpose is that of adapting Definition 21 and Definition 22 to the study of the asynchronous circuits that are real time, binary, non-deterministic processes and we consider the next

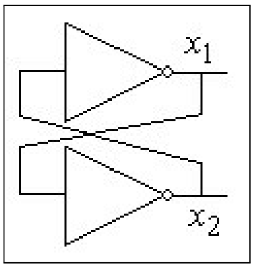

Figure 1:

Example 24 *In Figure 1 the logical gates compute the Boolean function* $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$,

$$\forall(\mu_1, \mu_2) \in \mathbf{B}^2, \Phi(\mu_1, \mu_2) = (\overline{\mu_2}, \overline{\mu_1}).$$

Let be $(t_k) \in Seq$ *and the function* $\rho : \mathbf{R} \to \mathbf{B}^2$,

$$\rho(t) = (1,0) \cdot \chi_{\{t_0\}}(t) \oplus (1,1) \cdot \chi_{\{t_1\}}(t) \oplus (0,1) \cdot \chi_{\{t_2\}}(t) \oplus$$

$$\oplus (1,0) \cdot \chi_{\{t_3\}}(t) \oplus (1,1) \cdot \chi_{\{t_4\}}(t) \oplus (0,1) \cdot \chi_{\{t_5\}}(t) \oplus ...$$

showing how Φ_1 *and* Φ_2 *are computed:* Φ_1 *at* t_0, Φ_1 *and* Φ_2 *at* t_1, Φ_2 *at* t_2, Φ_1 *at* t_3, Φ_1 *and* Φ_2 *at* t_4, Φ_2 *at* t_5,... *Denote by* $(\mu_1, \mu_2) \in \mathbf{B}^2$ *the initial state. We can see that*

at t_0 :	$(\Phi_1(\mu_1, \mu_2), \mu_2) = (\overline{\mu_2}, \mu_2)$
at t_1 :	$(\Phi_1(\overline{\mu_2}, \mu_2), \Phi_2(\overline{\mu_2}, \mu_2)) = (\overline{\mu_2}, \mu_2)$
at t_2 :	$(\overline{\mu_2}, \Phi_2(\overline{\mu_2}, \mu_2)) = (\overline{\mu_2}, \mu_2)$
at t_3 :	$(\Phi_1(\overline{\mu_2}, \mu_2), \mu_2) = (\overline{\mu_2}, \mu_2)$
at t_4 :	$(\Phi_1(\overline{\mu_2}, \mu_2), \Phi_2(\overline{\mu_2}, \mu_2)) = (\overline{\mu_2}, \mu_2)$
at t_5 :	$(\overline{\mu_2}, \Phi_2(\overline{\mu_2}, \mu_2)) = (\overline{\mu_2}, \mu_2)$

...

thus the behavior of the circuit is modeled by the function

$$x(t) = (\mu_1, \mu_2) \cdot \chi_{(-\infty, t_0)}(t) \oplus (\overline{\mu_2}, \mu_2) \cdot \chi_{[t_0, \infty)}(t).$$

3 Progressive sequences and progressive functions

Definition 25 *The sequence* $\alpha : \mathbf{N} \to \mathbf{B}^n, \alpha(k) \overset{not}{=} \alpha^k, k \in \mathbf{N}$ *is* ***progressive*** *if* $\forall i \in \{1, ..., n\}$, *the set*

$$\{k | k \in \mathbf{N}, \alpha_i^k = 1\}$$

is infinite. The set of the $\mathbf{N} \to \mathbf{B}^n$ *progressive sequences is denoted by* Π_n.

Definition 26 *The function $\rho : \mathbf{R} \to \mathbf{B}^n$ is called **progressive**, if $\alpha \in \Pi_n$ and $(t_k) \in Seq$ exist so that*

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{5}$$

The set of the progressive functions is denoted by P_n.

Remark 27 *In the previous two definitions, the condition that all the sets $\{k|k \in \mathbf{N}, \alpha_i^k = 1\}$ are infinite, $i = \overline{1,n}$ expresses the idea that the coordinates $\Phi_i, i = \overline{1,n}$ of some function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ are computed countably many times, thus their computation time is arbitrary, finite, possibly variable (depending on manufacturing fluctuations in delay related parameters, on the temperature, on the tension of the mains etc.) This was anticipated at Example 24.*

Some properties of invariance must be found in this context, of the form $..., \forall \rho \in P_n, ...$ meaning that for each manufactured instance of a design, for each admissible temperature and for each admissible tension of the mains, we restrict our attention to the information that is common to all of them. Such properties will be presented in Sections 22,...,28.

4 Boolean dynamical systems

Definition 28 *Let $\Phi : \mathbf{B}^n \to \mathbf{B}^n, \Phi = (\Phi_1, ..., \Phi_n)$ be a function (called sometimes vector field). For $\lambda \in \mathbf{B}^n, \lambda = (\lambda_1, ..., \lambda_n)$ we define the function $\Phi^\lambda : \mathbf{B}^n \to \mathbf{B}^n$,*

$$\forall \mu \in \mathbf{B}^n, \Phi^\lambda(\mu) = (\overline{\lambda_1} \cdot \mu_1 \oplus \lambda_1 \cdot \Phi_1(\mu), ..., \overline{\lambda_n} \cdot \mu_n \oplus \lambda_n \cdot \Phi_n(\mu))$$

*called Φ **at the power** λ, or the $\lambda-$**iterate** of Φ.*

Remark 29 *The role of λ in the previous definition is that of showing which coordinate Φ_i of Φ is computed: if $\lambda_i = 0$, then $\Phi_i^\lambda(\mu) = \mu_i$ and Φ_i is not computed, while if $\lambda_i = 1$, then $\Phi_i^\lambda(\mu) = \Phi_i(\mu)$ and Φ_i is computed, $i = \overline{1,n}$.*

Definition 30 *Let be $\alpha \in \Pi_n$. We define the functions $\Phi^{\alpha^0 ... \alpha^k} : \mathbf{B}^n \to \mathbf{B}^n, k \in \mathbf{N}$ by*

$$\forall k \in \mathbf{N}, \forall \mu \in \mathbf{B}^n, \Phi^{\alpha^0 ... \alpha^k \alpha^{k+1}}(\mu) = \Phi^{\alpha^{k+1}}(\Phi^{\alpha^0 ... \alpha^k}(\mu)). \tag{6}$$

Definition 31 *Let be $\rho \in P_n$,*

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{7}$$

*where $\alpha \in \Pi_n$ and $(t_k) \in Seq$. We define Φ **at the power** ρ, $\Phi^\rho : \mathbf{R} \to (\mathbf{B}^n)^{\mathbf{B}^n}$ by $\forall t \in \mathbf{R}, \forall \mu \in \mathbf{B}^n$,*

$$\Phi^\rho(t)(\mu) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \tag{8}$$

$$... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

Remark 32 *Suppose that (7) is true. A distinction should be made between* $\Phi^\rho(t)(\mu)$ *given by (8) and*

$$\Phi^{\rho(t)}(\mu) = \begin{cases} \Phi^{\alpha^k}(\mu), t = t_k \\ \mu, t \notin \{t_0, ..., t_k, ...\} \end{cases}.$$

Definition 33 *The couple* $\phi = (\mathbf{B}^n, (\Phi^\rho)_{\rho \in P_n})$ *is called* ***Boolean dynamical system****.* $\mathbf{B}^n$ *is called the* ***phase space****, or the* ***state space*** *and its points* $\mu \in \mathbf{B}^n$ *are called* ***phases****, or* ***states***[1]*. The function* Φ *is called the* ***generator function*** *of* ϕ *and* $\Phi^\rho, \rho \in P_n$ *are called the* ***computations*** *of* Φ*. The domain* $\mathbf{R}$ *of the computations* Φ^ρ *is the* ***time set*** *and* $t \in \mathbf{R}$ *is the* ***time parameter****.*

Remark 34 *We have given up the prefix 'semi' in the terminology of 'Boolean semi-dynamical system' because, in the theory of the asynchronous systems, the time set is always bounded from below (in the sense of the existence of the initial values of the states* $x(-\infty+0)$ *and of the initial time* t_0*); the possibility of allowing a time set which is unbounded from below corresponds to the prefix 'pseudo' in the terminology of 'Boolean dynamical pseudo-system' and we are not interested in the study of this concept at the moment.*

The major difference between ϕ *(Definition 33) and the usual semi-dynamical systems* γ *(Definition 22) comes from the fact that here the modelled processes are not deterministic, meaning that different ways that* Φ *may be computed exist and they are indicated by the power* ρ *of* Φ^ρ*. We conclude that the deterministic process in* γ *has been replaced by a family of deterministic processes in* ϕ*. The property (3) of no advance in time is replaced by*

$$\Phi^{(0,...,0)} = 1_{\mathbf{B}^n}$$

and the request (4) of advancing time is replaced by (6), (7), (8).

Definition 35 *For* $\mu \in \mathbf{B}^n$ *and* $\rho \in P_n$*, the function* $\Phi^\rho(\cdot)(\mu) : \mathbf{R} \to \mathbf{B}^n$ *is called the* $\rho-$***motion*** *of the point* μ*.*

Definition 36 *We define the* $\rho-$***orbit*** *(or the* $\rho-$***phase trajectory****)* ***of*** μ *in the next way:*

$$Orb_\rho(\mu) = \{\Phi^\rho(t)(\mu) | t \in \mathbf{R}\}.$$

Definition 37 *By definition, we call* ***integral curve*** *the graph*

$$G = \{(t, \Phi^\rho(t)(\mu)) | t \in \mathbf{R}\}.$$

[1] We abusively identify a function $x \in S^{(n)}$ normally called state with its values $\mu = x(t)$.

5 Regular autonomous asynchronous systems

Definition 38 *Let be* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$*. The set* $X_\Phi \subset S^{(n)}$,

$$X_\Phi = \{\mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

$$|\mu \in \mathbf{B}^n, \alpha \in \Pi_n, (t_k) \in Seq\}$$

is called the ***universal regular autonomous asynchronous system*** *that is* ***generated*** *by* Φ*. The function* Φ *is called the* ***generator function*** *of* X_Φ*.*

Definition 39 *A non-empty set* $X \in P^*(S^{(n)})$ *is called* ***regular autonomous asynchronous system*** *if* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *exists so that* $X \subset X_\Phi$*. If this inclusion holds, then the function* Φ *is called the* ***generator function*** *of* X*.*

Remark 40 *In Definitions 38, 39 the terminology is the following: 'universal' means maximal relative to the inclusion and 'regular' means the existence of a generator function* Φ*.*

While the system X_Φ *is associated to the Boolean dynamical system* $\phi = (\mathbf{B}^n, (\Phi^\rho)_{\rho \in P_n})$*, its subsystems* $X \subset X_\Phi$ *may result by requesting that the initial states* μ *run over a subset of* $\mathbf{B}^n$ *only (initial conditions), or perhaps* Φ^ρ *run over a subset of all the computations of* Φ *only (restrictions imposed on the computation time of the coordinate functions* $\Phi_i, i = \overline{1,n}$*).*

Notation 41 *Let be the set* $\Theta_0 \in P^*(\mathbf{B}^n)$*. Denote by* $X_\Phi^{\Theta_0} \subset X_\Phi$ *the system*

$$X_\Phi^{\Theta_0} = \{\mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

$$|\mu \in \Theta_0, \alpha \in \Pi_n, (t_k) \in Seq\}.$$

Remark 42 *In general the regular autonomous systems do not have a unique generator function:*

$$\exists \Phi, \Phi' : \mathbf{B}^n \to \mathbf{B}^n, \Phi \neq \Phi' \text{ and } X_\Phi \cap X_{\Phi'} \neq \emptyset.$$

Such a situation results in the intersection of the systems from Example 44 and Example 45 to follow.

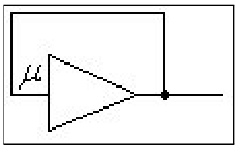

Figure 2:

6 Examples of regular autonomous asynchronous systems

Example 43 *Let us suppose that in Figure 1 the gates are identical, the computation time of the coordinate functions is equal to* 1*, the initial state is* $(0,0)$ *and the initial time instant is* $t_0 = 0$*. Then the model of this circuit is given by the autonomous system*

$$X = \{(1,1) \cdot \chi_{[1,2)\cup[3,4)\cup[5,6)\cup\ldots}\}.$$

Example 44 *We suppose that* $\Phi : \mathbf{B} \to \mathbf{B}$ *is the constant function:*

$$\exists \mu^0 \in \mathbf{B}, \forall \mu \in \mathbf{B}, \Phi(\mu) = \mu^0.$$

We identify the constant function $x \in S$ *with the constant* $\mu^0 \in \mathbf{B}$ *and we remark that*

$$X_\Phi = \{\mu^0\} \cup \{\overline{\mu^0} \cdot \chi_{(-\infty,t)} \oplus \mu^0 \cdot \chi_{[t,\infty)} | t \in \mathbf{R}\}.$$

Example 45 *For* $\Phi : \mathbf{B} \to \mathbf{B}$ *the identity function*

$$\forall \mu \in \mathbf{B}, \Phi(\mu) = \mu$$

we have

$$X_\Phi = \mathbf{B}$$

*(**B** has been identified with the set of the two constant 1-signals). The circuit was drawn in Figure 2.*

Example 46 *We define* $\Phi : \mathbf{B} \to \mathbf{B}$ *by*

$$\forall \mu \in \mathbf{B}, \Phi(\mu) = \overline{\mu}.$$

We infer that

$$X_\Phi = \{\mu \cdot \chi_{(-\infty,t_0)} \oplus \overline{\mu} \cdot \chi_{[t_0,t_1)} \oplus \mu \cdot \chi_{[t_1,t_2)} \oplus \overline{\mu} \cdot \chi_{[t_2,t_3)} \oplus \ldots | \mu \in \mathbf{B}, (t_k) \in Seq\}$$

and the circuit is the one from Figure 3.

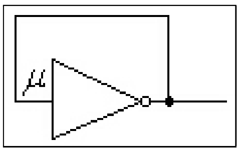

Figure 3:

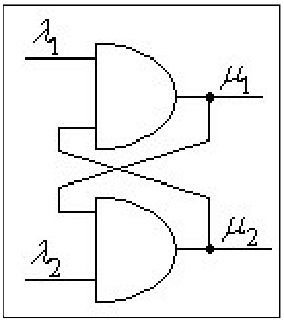

Figure 4:

Example 47 *We associate the function* $\Phi_\lambda : \mathbf{B}^2 \to \mathbf{B}^2$,

$$\forall \mu \in \mathbf{B}^2, \Phi_\lambda(\mu) = (\lambda_1\mu_2, \lambda_2\mu_1)$$

-where $\lambda \in \mathbf{B}^2$ *is a parameter- with the universal regular autonomous system* X_{Φ_λ}. *The states* $x \in X_{\Phi_\lambda}$ *model the behavior of the circuit from Figure 4 under the constant input* λ. *For example if* $\lambda = (0,0)$, *then*

$$X_{\Phi_{(0,0)}} = \{(\mu_1 \cdot \chi_{(-\infty,t_1)}, \mu_2 \cdot \chi_{(-\infty,t_2)})|\mu \in \mathbf{B}^2, t_1, t_2 \in \mathbf{R}\}.$$

7 Fixed points vs. final values

Remark 48 *The study of the fixed points is important because the final values of the states of the stable* $X \subset X_\Phi$ *systems are fixed points of* Φ *and the accessible fixed points of* Φ *are final values of the states of those systems.*

Theorem 49 *The next fixed point property is true*

$$\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n, \lim_{t\to\infty} \Phi^\rho(t)(\mu) = \mu' \Longrightarrow \Phi(\mu') = \mu'.$$

Proof. We take $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n$ and $\rho \in P_n$ arbitrarily with the property that $t' \in \mathbf{R}$ exists so that

$$\forall t \geq t', \Phi^\rho(t)(\mu) = \mu'. \tag{9}$$

We have the sequences $\alpha \in \Pi_n$ and $(t_k) \in Seq$ so that

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

and we can suppose that $t' = t_{k'}$ for some $k' \in \mathbf{N}$, thus from (9) we can write

$$\begin{gathered}\Phi^\rho(t_{k'})(\mu) = \Phi^{\alpha^0 ... \alpha^{k'}}(\mu) = \mu', \\ \forall k \geq 1, \Phi^\rho(t_{k'+k})(\mu) = \Phi^{\alpha^0 ... \alpha^{k'} \alpha^{k'+1} ... \alpha^{k'+k}}(\mu) = \\ = \Phi^{\alpha^{k'+1} ... \alpha^{k'+k}}(\Phi^{\alpha^0 ... \alpha^{k'}}(\mu)) = \Phi^{\alpha^{k'+1} ... \alpha^{k'+k}}(\mu') = \mu'.\end{gathered} \tag{10}$$

We define the sets $\Psi_{k'+1}, ..., \Psi_{k'+p} \subset \{1, ..., n\}$ in the following way:

$$\Psi_{k'+1} = \{i | i \in \{1, ..., n\}, \alpha_i^{k'+1} = 1\},$$

$$...$$

$$\Psi_{k'+p} = \{i | i \in \{1, ..., n\}, \alpha_i^{k'+p} = 1\},$$

$$\Psi_{k'+1} \cup ... \cup \Psi_{k'+p} = \{1, ..., n\}.$$

This is always possible and we have

$t \in [t_{k'+1}, t_{k'+2}) : \forall i \in \{1, ..., n\},$

$$x_i(t) = \begin{cases} \Phi_i(\mu'), i \in \Psi_{k'+1} \\ \mu'_i, i \in \{1, ..., n\} \setminus \Psi_{k'+1} \end{cases} \overset{(10)}{=} \mu'_i,$$

$t \in [t_{k'+2}, t_{k'+3}) : \forall i \in \{1, ..., n\},$

$$x_i(t) = \begin{cases} \Phi_i(\mu'), i \in \Psi_{k'+1} \cup \Psi_{k'+2} \\ \mu'_i, i \in \{1, ..., n\} \setminus (\Psi_{k'+1} \cup \Psi_{k'+2}) \end{cases} \overset{(10)}{=} \mu'_i,$$

$$...$$

$t \in [t_{k'+p}, \infty) : \forall i \in \{1, ..., n\},$

$$x_i(t) = \begin{cases} \Phi_i(\mu'), i \in \Psi_{k'+1} \cup ... \cup \Psi_{k'+p} \\ \mu'_i, i \in \{1, ..., n\} \setminus (\Psi_{k'+1} \cup ... \cup \Psi_{k'+p}) \end{cases} \overset{(10)}{=} \mu'_i.$$

It has resulted that

$$\forall i \in \{1, ..., n\}, \Phi_i(\mu') = \mu'_i.$$

■

Theorem 50 *We have* $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n,$

$$(\Phi(\mu') = \mu' \text{ and } \exists t' \in \mathbf{R}, \Phi^\rho(t')(\mu) = \mu') \Longrightarrow \lim_{t \to \infty} \Phi^\rho(t)(\mu) = \mu'.$$

Proof. We fix μ,$\mu' \in \mathbf{B}^n$, $\alpha \in \Pi_n$ and $(t_k) \in Seq$ arbitrarily so that

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

$$\Phi(\mu') = \mu'. \tag{11}$$

If $t' < t_0$, we obtain

$$\mu = \Phi^\rho(t')(\mu) \overset{hypothesis}{=} \mu'.$$

On the other hand, from (11) we get that

$$\Phi^{\alpha^0}(\mu) = ... = \Phi^{\alpha^0 ... \alpha^k}(\mu) = ... = \mu'$$

and the conclusion is fulfilled under the form

$$\forall t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu',$$

rest position.

We suppose now that $t' \geq t_0$, thus $k' \in \mathbf{N}$ exists with $t' \in [t_{k'}, t_{k'+1})$ and

$$\Phi^\rho(t')(\mu) = \Phi^{\alpha^0 ... \alpha^{k'}}(\mu) = \mu'.$$

But

$$\Phi^{\alpha^0 ... \alpha^{k'} \alpha^{k'+1}}(\mu) = \Phi^{\alpha^{k'+1}}(\Phi^{\alpha^0 ... \alpha^{k'}}(\mu)) = \Phi^{\alpha^{k'+1}}(\mu') \overset{(11)}{=} \mu',$$

$$...$$

$$\Phi^{\alpha^0 ... \alpha^{k'} \alpha^{k'+1} ... \alpha^{k'+k}}(\mu) = \Phi^{\alpha^{k'+1} ... \alpha^{k'+k}}(\Phi^{\alpha^0 ... \alpha^{k'}}(\mu)) = \Phi^{\alpha^{k'+1} ... \alpha^{k'+k}}(\mu') \overset{(11)}{=} \mu',$$

$$...$$

and the conclusion is

$$\forall t \geq t', \Phi^\rho(t)(\mu) = \mu'.$$

■

Remark 51 *Here are two other results that are inferred from Theorems 49 and 50:*

$$\forall x \in X_\Phi, \exists \lim_{t \to \infty} x(t) \Longrightarrow \Phi(\lim_{t \to \infty} x(t)) = \lim_{t \to \infty} x(t),$$

$$\forall x \in X_\Phi, \forall \mu' \in \mathbf{B}^n, (\Phi(\mu') = \mu' \text{ and } \exists t' \in \mathbf{R}, x(t') = \mu') \Longrightarrow \lim_{t \to \infty} x(t) = \mu'.$$

8 Nullclins and fixed points

Definition 52 *Let be the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$. For any $i \in \{1, ..., n\}$, the **nullclins** of Φ are the sets*

$$NC_i = \{\mu | \mu \in \mathbf{B}^n, \Phi_i(\mu) = \mu_i\}.$$

*If $\mu \in NC_i$, then the coordinate i is said to be **not excited**, or **not enabled**, or **stable** and otherwise it is called **excited**, or **enabled**, or **unstable**.*

Theorem 53 *The following statements are equivalent for $\mu \in \mathbf{B}^n$:*
a) $\exists \rho \in P_n, Orb_\rho(\mu) = \{\mu\}$,
b) $\exists \rho \in P_n, \forall t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu$,
c) $\Phi(\mu) = \mu$,
d) $\mu \in NC_1 \cap ... \cap NC_n$.

Proof. a)$\Longleftrightarrow$b) and c)$\Longleftrightarrow$d) are obvious from the way that $Orb_\rho(\mu)$ and $NC_1, ..., NC_n$ were defined.

b)$\Longrightarrow$c) Let be $\alpha \in \Pi_n$ and $(t_k) \in Seq$ with the property that

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{12}$$

and b) is fulfilled. We obtain

$$\Phi^\rho(t)(\mu) = \tag{13}$$

$$= \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... = \mu$$

$$\Longleftrightarrow \Phi^{\alpha^0}(\mu) = ... = \Phi^{\alpha^0 ... \alpha^k}(\mu) = ... = \mu.$$

We denote

$$\Psi_0 = \{i | i \in \{1, ..., n\}, \alpha_i^0 = 1\}, ..., \Psi_k = \{i | i \in \{1, ..., n\}, \alpha_i^k = 1\}, ...$$

and we infer that

$$\forall k \in \mathbf{N}, \forall i \in \Psi_0 \cup ... \cup \Psi_k, \Phi_i(\mu) = \mu_i.$$

The existence of $k \in \mathbf{N}$ such that $\Psi_0 \cup ... \cup \Psi_k = \{1, ..., n\}$ shows the validity of c).

c)$\Longrightarrow$b) Let $\alpha \in \Pi_n$ and $(t_k) \in Seq$ be arbitrary and we define $\rho \in P_n$ by (12). It is shown by induction on $k \in \mathbf{N}$ that

$$\Phi^{\alpha^0}(\mu) = ... = \Phi^{\alpha^0 ... \alpha^k}(\mu) = ... = \mu$$

wherefrom we get (see (13)) the truth of

$$\forall t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu$$

for arbitrary $\rho \in P_n$. b) is true. ■

Notation 54 *The set of the fixed points (also called points of equilibrium) of Φ is denoted by Eq.*

9 The speed

Definition 55 *For any $\mu \in \mathbf{B}^n$ and $\rho \in P_n$, we define the* ***speed of the*** $\rho-$***motion of*** μ, $v^\rho(\cdot)(\mu) : \mathbf{R} \to \mathbf{B}^n$ *in the following manner*

$$\forall t \in \mathbf{R}, v^\rho(t)(\mu) = \Phi^\rho(t-0)(\mu) \oplus \Phi^\rho(t)(\mu).$$

Remark 56 *We infer from the previous definition that if*

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...,$$

then

$$v^\rho(t)(\mu) = D\Phi^\rho(t)(\mu) =$$

$$= (\mu \oplus \Phi^{\alpha^0}(\mu)) \cdot \chi_{\{t_0\}}(t) \oplus (\Phi^{\alpha^0}(\mu) \oplus \Phi^{\alpha^0\alpha^1}(\mu)) \cdot \chi_{\{t_1\}}(t) \oplus ...$$

$$... \oplus (\Phi^{\alpha^0...\alpha^{k-1}}(\mu) \oplus \Phi^{\alpha^0...\alpha^k}(\mu)) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

10 The relation between equations and dynamical systems

Theorem 57 *We consider the point $\mu \in \mathbf{B}^n$ and the function $\rho \in P_n$.*

i) The equations

$$\begin{cases} x(-\infty+0) = \mu \\ Dx(t) = v^\rho(t)(\mu) \end{cases},$$

$$\begin{cases} x(-\infty+0) = \mu \\ x(t) = \Phi^{\rho(t)}(x(t-0)) \end{cases}$$

have both the unique solution

$$x(t) = \Phi^\rho(t)(\mu).$$

ii) μ is a fixed point of $\Phi \Longleftrightarrow \forall t \in \mathbf{R}, v^\rho(t)(\mu) = 0$.

Proof. Let be $\mu \in \mathbf{B}^n, \alpha \in \Pi_n$ and $(t_k) \in Seq$. We denote

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{14}$$

We search the solutions of the two equations under the form

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... \tag{15}$$

where the indeterminates are $x(t_k) \in \mathbf{B}^n, k \in \mathbf{N}$. As we know, we have

$$\Phi^\rho(t)(\mu) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \tag{16}$$

$$... \oplus \Phi^{\alpha^0...\alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

$$v^{\rho}(t)(\mu)=(\mu\oplus\Phi^{\alpha^0}(\mu))\cdot\chi_{\{t_0\}}(t)\oplus(\Phi^{\alpha^0}(\mu)\oplus\Phi^{\alpha^0\alpha^1}(\mu))\cdot\chi_{\{t_1\}}(t)\oplus... \quad (17)$$

$$...\oplus(\Phi^{\alpha^0...\alpha^{k-1}}(\mu)\oplus\Phi^{\alpha^0...\alpha^k}(\mu))\cdot\chi_{\{t_k\}}(t)\oplus...$$

We solve the first equation, where

$$Dx(t)=(\mu\oplus x(t_0))\cdot\chi_{\{t_0\}}(t)\oplus(x(t_0)\oplus x(t_1))\cdot\chi_{\{t_1\}}(t)\oplus... \quad (18)$$

$$\oplus(x(t_{k-1})\oplus x(t_k))\cdot\chi_{\{t_k\}}(t)\oplus...$$

and we infer from (17) and (18):

$$\mu\oplus x(t_0)=\mu\oplus\Phi^{\alpha^0}(\mu),$$

$$x(t_0)\oplus x(t_1)=\Phi^{\alpha^0}(\mu)\oplus\Phi^{\alpha^0\alpha^1}(\mu),$$

$$...$$

$$x(t_{k-1})\oplus x(t_k)=\Phi^{\alpha^0...\alpha^{k-1}}(\mu)\oplus\Phi^{\alpha^0...\alpha^k}(\mu),$$

$$...$$

In other words

$$x(t_0)=\Phi^{\alpha^0}(\mu),$$

$$x(t_1)=\Phi^{\alpha^0\alpha^1}(\mu),$$

$$...$$

$$x(t_k)=\Phi^{\alpha^0...\alpha^k}(\mu),$$

$$...$$

thus $\Phi^{\rho}(t)(\mu)$ is a solution of the first equation.

We solve the second equation and we take into account the fact that

$$x(t-0)=\mu\cdot\chi_{(-\infty,t_0]}(t)\oplus x(t_0)\cdot\chi_{(t_0,t_1]}(t)\oplus...\oplus x(t_k)\cdot\chi_{(t_k,t_{k+1}]}(t)\oplus...$$

$$\Phi^{\rho(t)}(x(t-0))=\begin{cases}\Phi^{\alpha^k}(x(t-0)),t=t_k\\ x(t-0),t\notin\{t_0,...,t_k,...\}\end{cases}.$$

We have

$t<t_0$: $\quad x(t)=x(t-0)=x(-\infty+0)=\mu,$

$t=t_0$: $\quad x(t_0)=\Phi^{\alpha^0}(x(t_0-0))=\Phi^{\alpha^0}(\mu),$

$t\in(t_0,t_1)$: $\quad x(t)=x(t-0)=\Phi^{\alpha^0}(\mu),$

$t=t_1$: $\quad x(t_1)=\Phi^{\alpha^1}(x(t_1-0))=\Phi^{\alpha^1}(\Phi^{\alpha^0}(\mu))=\Phi^{\alpha^0\alpha^1}(\mu),$

$t\in(t_1,t_2)$: $\quad x(t)=x(t-0)=\Phi^{\alpha^0\alpha^1}(\mu),$

$$...$$

$t=t_k$: $\quad x(t_k)=\Phi^{\alpha^k}(x(t_k-0))=\Phi^{\alpha^k}(\Phi^{\alpha^0...\alpha^{k-1}}(\mu))=\Phi^{\alpha^0...\alpha^k}(\mu),$

$t \in (t_k, t_{k+1}) : x(t) = x(t-0) = \Phi^{\alpha^0 ... \alpha^k}(\mu),$

$$...$$

The fact that $\Phi^\rho(t)(\mu)$ satisfies the second equation was proved.

The uniqueness of the solution is proved for both equations like this. We suppose against all reason that two distinct solutions x, x' exist. Then we observe that $\forall t < t_0$, $x(t) = x'(t) = \mu$, thus $t_1 \geq t_0$ should exist such that $\forall t < t_1, x(t) = x'(t)$ and $x(t_1) \neq x'(t_1)$. This supposition gives a contradiction in both cases, meaning that the solution is unique.

The statement ii) is a consequence of the fact that $\Phi(\mu) = \mu \Longleftrightarrow \mu = \Phi^{\alpha^0}(\mu) = \Phi^{\alpha^0\alpha^1}(\mu) = ...$ $\blacksquare$

Remark 58 *We can associate to any of the equations from Theorem 57, when μ runs in $\mathbf{B}^n$ and ρ runs in P_n, the dynamical system $\phi = (\mathbf{B}^n, (\Phi^\rho)_{\rho \in P_n})$.*

11 Invariant sets in the space of the phases

Definition 59 *The set $A \in P^*(\mathbf{B}^n)$ is $\rho-$**invariant**, $\rho \in P_n$ if*

$$\forall \mu \in A, \forall t \in \mathbf{R}, \Phi^\rho(t)(\mu) \in A. \tag{19}$$

Other definitions of invariance of A are the following:

$$\forall \mu \in A, \exists \rho' \in P_n, \forall t \in \mathbf{R}, \Phi^{\rho'}(t)(\mu) \in A, \tag{20}$$

$$\forall \mu \in A, \forall \rho' \in P_n, \forall t \in \mathbf{R}, \Phi^{\rho'}(t)(\mu) \in A, \tag{21}$$

$$\forall \mu \in A, \forall \lambda \in \mathbf{B}^n, \Phi^\lambda(\mu) \in A. \tag{22}$$

Theorem 60 *The following implications hold:*

$$(22) \Longleftrightarrow (21) \Longrightarrow (19) \Longrightarrow (20).$$

Proof. We prove (21)$\Longleftrightarrow$(22), because the other implications are obvious.

(21)$\Longrightarrow$(22) Let $\mu \in A, \lambda \in \mathbf{B}^n$ and the sequence $\rho'' \in P_n$ be arbitrary,

$$\rho''(t) = \alpha''^0 \cdot \chi_{\{t''_0\}}(t) \oplus ... \oplus \alpha''^k \cdot \chi_{\{t''_k\}}(t) \oplus ...$$

with $\alpha'' \in \Pi_n$ and $(t''_k) \in Seq$. We define

$$\rho'(t) = \lambda \cdot \chi_{\{t_0\}}(t) \oplus \alpha''^0 \cdot \chi_{\{t_0 + t''_0\}}(t) \oplus ... \oplus \alpha''^k \cdot \chi_{\{t_0 + t''_k\}}(t) \oplus ...$$

where $t_0 \in \mathbf{R}$ is arbitrary and we can see that $\rho' \in P_n$. (21) implies $\Phi^\lambda(\mu) = \Phi^{\rho'}(t_0)(\mu) \in A$.

(22)$\Longrightarrow$(21) Let $\mu \in A$ and $\rho' \in P_n$,

$$\rho'(t) = \alpha'^0 \cdot \chi_{\{t'_0\}}(t) \oplus ... \oplus \alpha'^k \cdot \chi_{\{t'_k\}}(t) \oplus ...$$

be arbitrary, with $\alpha' \in \Pi_n$, $(t'_k) \in Seq$. (22) implies that

$t < t'_0$: $\qquad \Phi^{\rho'}(t)(\mu) = \mu \in A$,

$t \in [t'_0, t'_1)$: $\qquad \Phi^{\rho'}(t)(\mu) = \Phi^{\alpha'^0}(\mu) \in A$,

...

$t \in [t'_k, t'_{k+1})$: $\quad \Phi^{\alpha'^0 ... \alpha'^{k-1}}(\mu) \in A$ due to the hypothesis of the induction and we have $\Phi^{\rho'}(t)(\mu) = \Phi^{\alpha'^k}(\Phi^{\alpha'^0 ... \alpha'^{k-1}}(\mu)) \in A$,

...

■

12 Examples of invariant sets

Example 61 *Let be* $\Phi : \mathbf{B}^2 \to \mathbf{B}^2, \forall \mu \in \mathbf{B}^2, \Phi(\mu_1, \mu_2) = (\overline{\mu_1}, \overline{\mu_2})$ *and* $\rho(t) = (1,1) \cdot \chi_{\{0,1,2,...\}}(t)$. *The set* $A = \{(0,1),(1,0)\}$ *is* $\rho-$*invariant i.e. it satisfies* (19):

$$\Phi^\rho(t)(0,1) = (0,1) \cdot \chi_{(-\infty,0)}(t) \oplus (1,0) \cdot \chi_{[0,1)}(t) \oplus$$

$$\oplus (0,1) \cdot \chi_{[1,2)}(t) \oplus (1,0) \cdot \chi_{[2,3)}(t) \oplus ...$$

$$\Phi^\rho(t)(1,0) = (1,0) \cdot \chi_{(-\infty,0)}(t) \oplus (0,1) \cdot \chi_{[0,1)}(t) \oplus$$

$$\oplus (1,0) \cdot \chi_{[1,2)}(t) \oplus (0,1) \cdot \chi_{[2,3)}(t) \oplus ...$$

Similarly, $A = \{(0,0),(1,1)\}$ *satisfies the same invariance property.*

Example 62 *Let be* $\mu \in \mathbf{B}^n$. *The set* $\bigcup_{\rho \in P_n} Orb_\rho(\mu)$ *is invariant in the sense of the satisfaction of (21) and in order to see this we take an arbitrary vector* $\mu' \in \bigcup_{\rho \in P_n} Orb_\rho(\mu)$, *for which we have two possibilities:*

a) $\mu' = \mu$

In this situation (21) is obviously fulfilled.

b) $\mu' \neq \mu$

In this case $\alpha \in \Pi_n$, $(t_k) \in Seq$ *and* $k' \in \mathbf{N}$ *exist such that*

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

$$\mu' = \Phi^\rho(t_{k'})(\mu) = \Phi^{\alpha^0 ... \alpha^{k'}}(\mu).$$

If $\rho' \in P_n$ is arbitrary,

$$\rho'(t) = \alpha'^0 \cdot \chi_{\{t'_0\}}(t) \oplus ... \oplus \alpha'^k \cdot \chi_{\{t'_k\}}(t) \oplus ...$$

$\alpha' \in \Pi_n, (t'_k) \in Seq$, then the sequence

$$\gamma(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^{k'} \cdot \chi_{\{t_{k'}\}}(t) \oplus$$

$$\oplus \alpha'^0 \cdot \chi_{\{t_{k'}+t'_0\}}(t) \oplus ... \oplus \alpha'^k \cdot \chi_{\{t_{k'}+t'_k\}}(t) \oplus ...$$

belongs to P_n and we have for $t \in \mathbf{R}$ the following possibilities:

b.1) $t < t'_0$ when $\Phi^{\rho'}(t)(\mu') = \mu' \in \bigcup\limits_{\rho \in P_n} Orb_\rho(\mu)$;

b.2) $t \geq t'_0$ when $k'' \in \mathbf{N}$ exists such that $t \in [t'_{k''}, t'_{k''+1})$ and

$$\Phi^{\rho'}(t)(\mu') = \Phi^{\rho'}(t'_{k''})(\Phi^{\rho}(t_{k'})(\mu)) = \Phi^{\rho'}(t'_{k''})(\Phi^{\alpha^0 ... \alpha^{k'}}(\mu)) =$$

$$= \Phi^{\alpha'^0 ... \alpha'^{k''}}(\Phi^{\alpha^0 ... \alpha^{k'}}(\mu)) = \Phi^{\alpha^0 ... \alpha^{k'} \alpha'^0 ... \alpha'^{k''}}(\mu) = \Phi^{\gamma}(t_{k'} + t'_{k''})(\mu)$$

thus $\Phi^{\rho'}(t)(\mu') \in Orb_\gamma(\mu) \subset \bigcup\limits_{\rho \in P_n} Orb_\rho(\mu)$.

Example 63 *We show that Eq is invariant in the sense of satisfaction of (22). Indeed, let be $\mu \in Eq$ meaning that*

$$\Phi(\mu) = \mu \tag{23}$$

is true and we take arbitrarily some $\lambda \in \mathbf{B}^n$. Because

$$\forall i \in \{1, ..., n\}, \Phi_i^\lambda(\mu) = \begin{cases} \Phi_i(\mu), if\ \lambda_i = 1 \\ \mu_i, if\ \lambda_i = 0 \end{cases} = \mu_i, \tag{24}$$

we conclude that

$$\Phi(\Phi^\lambda(\mu)) \overset{(24)}{=} \Phi(\mu) \overset{(23)}{=} \mu \overset{(24)}{=} \Phi^\lambda(\mu),$$

thus $\Phi^\lambda(\mu) \in Eq$.

13 Attraction

Definition 64 *Let be $A, B \in P^*(\mathbf{B}^n)$ and the function $\rho \in P_n$. We say that A is $\rho-$**attractive for** B and that the points of B are $\rho-$**attracted by the set** A if*

$$\forall \mu \in B, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^\rho(t')(\mu) \in A, \tag{25}$$

*i.e. A contains all the cyclic values of $\Phi^\rho(t)(\mu), \mu \in B$. Similarly, A is **attractive for** B and the points of B are **attracted by** A if one of the non-equivalent properties*

$$\forall \mu \in B, \exists \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A, \tag{26}$$

$$\forall \mu \in B, \forall \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A \tag{27}$$

holds.

Remark 65 *The implications*

$$(27) \Longrightarrow (25) \Longrightarrow (26)$$

hold. On the other hand if $A \in P^*(\mathbf{B}^n)$ *is* $\rho-$*invariant ((*19*) is fulfilled) then it is* $\rho-$*attractive for itself ((*25*) is true with* $B = A$*).*

Here is the special case of Definition 64 when $B = \{\mu\}$ *:*

$$\exists t \in \mathbf{R}, \forall t' \geq t, \Phi^\rho(t')(\mu) \in A,$$

$$\exists \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A,$$
$$\forall \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A.$$

14 Limit cycles

Definition 66 *The set* B *and the function* ρ *are given. The* $\rho-$***limit cycle*** *(*$\rho-$***limit set****)* ***of*** B *is given by:*

$$LC_B^\rho = \{\mu' | \mu' \in \mathbf{B}^n, \exists \mu \in B, \forall t \in \mathbf{R}, \exists t' > t, \Phi^\rho(t')(\mu) = \mu'\}.$$

Each $\mu' \in LC_B^\rho$ *is called* $\rho-$***cyclic****, or* $\rho-$***limit point of*** B*. Similarly, the* ***superior*** *and the* ***inferior limit cycle (limit set)*** *of* B *are defined by*

$$\overline{LC}_B = \{\mu' | \mu' \in \mathbf{B}^n, \exists \mu \in B, \exists \rho' \in P_n, \forall t \in \mathbf{R}, \exists t' > t, \Phi^{\rho'}(t')(\mu) = \mu'\},$$

$$\underline{LC}_B = \{\mu' | \mu' \in \mathbf{B}^n, \exists \mu \in B, \forall \rho' \in P_n, \forall t \in \mathbf{R}, \exists t' > t, \Phi^{\rho'}(t')(\mu) = \mu'\}.$$

Remark 67 *The limit cycles are the sets of the cyclic values of the motions* $\Phi^\rho(t)(\mu)$ *and the following inclusions*

$$\underline{LC}_B \subset LC_B^\rho \subset \overline{LC}_B$$

are true.

We have the special case at Definition 66, when $B = \{\mu\}$ *:*

$$LC_\mu^\rho = \{\mu' | \mu' \in \mathbf{B}^n, \forall t \in \mathbf{R}, \exists t' > t, \Phi^\rho(t')(\mu) = \mu'\},$$

$$\overline{LC}_\mu = \{\mu' | \mu' \in \mathbf{B}^n, \exists \rho' \in P_n, \forall t \in \mathbf{R}, \exists t' > t, \Phi^{\rho'}(t')(\mu) = \mu'\},$$
$$\underline{LC}_\mu = \{\mu' | \mu' \in \mathbf{B}^n, \forall \rho' \in P_n, \forall t \in \mathbf{R}, \exists t' > t, \Phi^{\rho'}(t')(\mu) = \mu'\}.$$

Theorem 68 *Let be* $B \in P^*(\mathbf{B}^n)$*,* $\rho \in P_n$ *and we consider the Definition 64, property (25). Then the set* LC_B^ρ *is* $\rho-$*attractive for* B *and any* $A \in P^*(\mathbf{B}^n)$ *which is* $\rho-$*attractive for* B *fulfills* $LC_B^\rho \subset A$*. Similar properties hold for the properties (26), (27) and for the sets* $\overline{LC}_B$*,* $\underline{LC}_B$*.*

Example 69 *The function* $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$ *defined by the following table:*

(μ_1, μ_2)	Φ
$(0,0)$	$(0,1)$
$(0,1)$	$(1,1)$
$(1,0)$	$(1,1)$
$(1,1)$	$(0,1)$

has the interesting property that $\forall \mu \in \mathbf{B}^2, \forall \rho \in P_n$,

$$LC^{\rho}_{\mu} = \{(0,1),(1,1)\}.$$

Theorem 70 *Let be* $\mu \in \mathbf{B}^n$ *and* $\rho \in P_n$. *We have that:*
a) $LC^{\rho}_{\mu} \subset Orb_{\rho}(\mu)$,
b) $LC^{\rho}_{\mu} \neq \emptyset$,
c) $LC^{\rho}_{\mu} = \{\mu'\} \Longrightarrow \lim\limits_{t \to \infty} \Phi^{\rho}(t)(\mu) = \mu'$,
d) $\exists t_0 \in \mathbf{R}, \forall t \geq t_0, \Phi^{\rho}(t)(\mu) \in LC^{\rho}_{\mu}$,
e) The set LC^{ρ}_{μ} *is invariant in the sense of satisfying (20).*

Proof. d) This is true from the definition of LC^{ρ}_{μ}.

e) Let $t_0 \in \mathbf{R}$ be the number that makes d) be true and we take some arbitrary $\mu' \in LC^{\rho}_{\mu}$. As all the points of LC^{ρ}_{μ} are $\rho-$cyclic values of $\Phi^{\rho}(\cdot)(\mu)$, we have

$$\exists t' > t_0, \Phi^{\rho}(t')(\mu) = \mu'.$$

We can see that the function

$$\rho'(t) = \rho(t) \cdot \chi_{(t',\infty)}(t)$$

is progressive and we infer that

$$\Phi^{\rho'}(t)(\mu') = \begin{cases} \mu', t \leq t' \\ \Phi^{\rho}(t)(\mu), t > t' \end{cases} \in LC^{\rho}_{\mu}.$$

■

15 The evolutions of a system

Definition 71 *We have the following terminology, given by Moisil and Gavrilov [3], [4]:*
a) $Orb_{\rho}(\mu) \setminus LC^{\rho}_{\mu} \neq \emptyset,\ |LC^{\rho}_{\mu}| > 1$
Moisil: ***finally cyclic evolution,***
Gavrilov: ***successively repeated evolution;***
b) $Orb_{\rho}(\mu) \setminus LC^{\rho}_{\mu} \neq \emptyset,\ |LC^{\rho}_{\mu}| = 1$
Moisil: ***finally stabilized evolution,***

Gavrilov: ***successive evolution;***

c) $Orb_\rho(\mu) = LC^\rho_\mu,\ |LC^\rho_\mu| > 1$

Moisil: ***cyclic evolution,***

Gavrilov: ***repeated evolution;***

d) $Orb_\rho(\mu) = LC^\rho_\mu,\ |LC^\rho_\mu| = 1$

Moisil: ***rest position*** *(or* ***stable position****).*

Remark 72 *We see that both Moisil and Gavrilov avoid referring to periodicity or maybe to pseudo-periodicity; for them the evolution of a system is just cyclic or repeated. On the other hand in cases b), d) when* $|LC^\rho_\mu| = 1$*, the* $\rho-$*limit cycle* LC^ρ_μ *consists of a point* μ' *that is a final value of* $\Phi^\rho(t)(\mu)$*, thus a fixed point of* Φ *(see Theorem 49).*

16 Stable manifold

Definition 73 *Let be* $A \in P^*(\mathbf{B}^n)$ *and* $\rho \in P_n$*. The* $\rho-$***stable manifold***[2] ***of*** A*, or the* ***kingdom*** *(the* ***basin****)* ***of*** $\rho-$***attraction of*** A *is by definition the set*

$$M^\rho_s(A) = \{\mu | \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^\rho(t')(\mu) \in A\}. \tag{28}$$

Two versions of this definition are the following:

$$\overline{M}_s(A) = \{\mu | \mu \in \mathbf{B}^n, \exists \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A\}, \tag{29}$$

$$\underline{M}_s(A) = \{\mu | \mu \in \mathbf{B}^n, \forall \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A\} \tag{30}$$

giving the ***superior*** *and the* ***inferior stable manifold of*** A*.*

Remark 74 *The stable manifolds of A are the sets of points which are attracted by A and the following inclusions hold*

$$\underline{M}_s(A) \subset M^\rho_s(A) \subset \overline{M}_s(A).$$

We have the special case at Definition 73, when $A = \{\mu^0\}$ *:*

$$M^\rho_s(\mu^0) = \{\mu | \mu \in \mathbf{B}^n, \lim_{t\to\infty} \Phi^\rho(t)(\mu) = \mu^0\},$$

$$\overline{M}_s(\mu^0) = \{\mu | \mu \in \mathbf{B}^n, \exists \rho' \in P_n, \lim_{t\to\infty} \Phi^{\rho'}(t)(\mu) = \mu^0\},$$

$$\underline{M}_s(\mu^0) = \{\mu | \mu \in \mathbf{B}^n, \forall \rho' \in P_n, \lim_{t\to\infty} \Phi^{\rho'}(t)(\mu) = \mu^0\}.$$

Theorem 75 *Let* $A \in P^*(\mathbf{B}^n), \rho \in P_n$ *be given and we consider the Definition 73, property (28). We have that the points of* $M^\rho_s(A)$ *are* $\rho-$*attracted by A and for any set* $B \in P^*(\mathbf{B}^n)$ *whose points are* $\rho-$*attracted by A we infer* $B \subset M^\rho_s(A)$*. Similar statements are true for the properties (29), (30) and the sets* $\overline{M}_s(A), \underline{M}_s(A)$*.*

[2] The word 'manifold' is traditional and it was borroughed from the literature by analogy; in our case it has not a precise meaning, since the manifolds were not defined in the binary context.

17 Total and partial attraction

Definition 76 *If $M_s^\rho(A) \neq \emptyset$, then the set A is called $\rho-$**attractive**. In this case we have the possibilities:*

*a) $M_s^\rho(A) = \mathbf{B}^n$, when A is called **totally** $\rho-$**attractive***

*b) $M_s^\rho(A) \neq \mathbf{B}^n$, when A is called **partially** $\rho-$**attractive**.*

*By replacing $M_s^\rho(A)$ with $\overline{M}_s(A)$ (with $\underline{M}_s(A)$), we obtain the **superior** (the **inferior**) **attractive**, **totally attractive** and **partially attractive** sets.*

Remark 77 *We have the special case at Definition 76, when $A = \mu^0$. If $M_s^\rho(\mu^0) \neq \emptyset$, then μ^0 is a point of $\rho-$attraction:*

$$\exists \mu \in \mathbf{B}^n, \lim_{t \to \infty} \Phi^\rho(t)(\mu) = \mu^0$$

and we have the possibilities: $M_s^\rho(\mu^0) = \mathbf{B}^n$, when μ^0 is totally $\rho-$attractive, respectively $M_s^\rho(\mu^0) \neq \mathbf{B}^n$, when μ^0 is partially $\rho-$attractive.

The situation is similar by replacing $M_s^\rho(\mu^0)$ with $\overline{M}_s(\mu^0)$ and $\underline{M}_s(\mu^0)$.

18 Equivalencies

Definition 78 *Let be the generator functions $\Phi, \Psi : \mathbf{B}^n \to \mathbf{B}^n$. We say that the Boolean dynamical systems $\phi = (\mathbf{B}^n, (\Phi^\rho)_{\rho \in P_n})$ and $\psi = (\mathbf{B}^n, (\Psi^\rho)_{\rho \in P_n})$ are **equivalent** if a bijection $H : \mathbf{B}^n \to \mathbf{B}^n$ exists so that the following diagram is commutative*

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Phi}{\to} & \mathbf{B}^n \\ H \downarrow & & \downarrow H \\ \mathbf{B}^n & \overset{\Psi}{\to} & \mathbf{B}^n \end{array}$$

*If this is true, we say that H **transforms the generator function** Φ **in the generator function** Ψ.*

Remark 79 *This definition concerning the equivalence of the Boolean dynamical systems refers to a change of the system of coordinates.*

Lemma 80 *If the function $h : \mathbf{R} \to \mathbf{R}$ is bijective, continuous and strictly increasing, then h^{-1} has the same properties.*

Proof. [3]In [2], page 233 it is shown that if $h : I \to \mathbf{R}$ is continuous and strictly monotonous on the interval $I \subset \mathbf{R}$ (the choice $I = \mathbf{R}$ is possible), then it has a continuous inverse. Obviously, if $h : \mathbf{R} \to \mathbf{R}$ is bijective and strictly monotonous, its inverse is strictly monotonous, with the same monotony like h.

The conclusion is that if h is bijective, continuous and strictly increasing, its inverse has the same properties. ■

[3]The proof of this Lemma was suggested to us by professor Sorin Gal.

Definition 81 *Let be $\mu \in \mathbf{B}^n$ and the functions $\rho, \rho' \in P_n$. The equations*

$$\begin{cases} x(-\infty+0) = \mu \\ x(t) = \Phi^{\rho(t)}(x(t-0)) \end{cases}, \tag{31}$$

$$\begin{cases} y(-\infty+0) = \mu \\ y(t) = \Phi^{\rho'(t)}(y(t-0)) \end{cases} \tag{32}$$

and the motions $\Phi^{\rho}(\cdot)(\mu)$ and $\Phi^{\rho'}(\cdot)(\mu)$ are ***equivalent*** *if the bijective, continuous strictly increasing function $h : \mathbf{R} \to \mathbf{R}$ exists so that*

$$\Phi^{\rho'}(t)(\mu) = \Phi^{\rho}(h(t))(\mu). \tag{33}$$

Remark 82 *This definition states that the solutions $\Phi^{\rho}(\cdot)(\mu)$, $\Phi^{\rho'}(\cdot)(\mu)$ of (31),(32) are equivalent if they are equal functions regardless the time flow, which is given by t for $\Phi^{\rho'}(\cdot)(\mu)$ and by $h(t)$ for $\Phi^{\rho}(\cdot)(\mu)$. Lemma 80 guarantees that Definition 81 is indeed that of an equivalence relation. We have the sufficient condition $\rho' = \rho \circ h$ in order that (33) is true:*

Theorem 83 *Let be $\mu \in \mathbf{B}^n, \rho \in P_n$ and $h : \mathbf{R} \to \mathbf{R}$ bijective, continuous strictly increasing. Then*

$$\Phi^{\rho \circ h}(t)(\mu) = \Phi^{\rho}(h(t))(\mu).$$

Proof. We take the sequences $\alpha \in \Pi_n$ and $(t_k) \in Seq$ so that

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

and we remark the truth of the following statements

$$h(t) \in (-\infty, t_0) \Longleftrightarrow t \in (-\infty, h^{-1}(t_0)),$$

$$h(t) \in [t_0, t_1) \Longleftrightarrow t \in [h^{-1}(t_0), h^{-1}(t_1)),$$

$$...$$

$$h(t) \in [t_k, t_{k+1}) \Longleftrightarrow t \in [h^{-1}(t_k), h^{-1}(t_{k+1})),$$

$$...$$

Because

$$(\rho \circ h)(t) = \rho(h(t)) = \alpha^0 \cdot \chi_{\{t_0\}}(h(t)) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(h(t)) \oplus ...$$

$$= \alpha^0 \cdot \chi_{\{h^{-1}(t_0)\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{h^{-1}(t_k)\}}(t) \oplus ...$$

we infer that for arbitrary $\mu \in \mathbf{B}^n$ we can write

$$\Phi^{\rho \circ h}(t)(\mu) =$$

$$= \mu \cdot \chi_{(-\infty, h^{-1}(t_0))}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[h^{-1}(t_0), h^{-1}(t_1))}(t) \oplus ...$$

$$... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[h^{-1}(t_k), h^{-1}(t_{k+1}))}(t) \oplus ...$$

$$= \mu \cdot \chi_{(-\infty, t_0)}(h(t)) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0, t_1)}(h(t)) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k, t_{k+1})}(h(t)) \oplus ...$$

$$= \Phi^{\rho}(h(t))(\mu).$$

■

Definition 84 *Let be $\mu \in \mathbf{B}^n$ and $\rho, \rho' \in P_n$. The equations (31), (32) and the motions $\Phi^{\rho}(\cdot)(\mu)$, $\Phi^{\rho'}(\cdot)(\mu)$ are* ***equivalent*** *if*

$$LC_{\mu}^{\rho} = LC_{\mu}^{\rho'}.$$

Remark 85 *The motions $\Phi^{\rho}(\cdot)(\mu)$, $\Phi^{\rho'}(\cdot)(\mu)$ are equivalent conformably to Definition 84 if they start from the same initial value μ and they reach the same limit cycle. The equivalence class of $\Phi^{\rho}(\cdot)(\mu)$ has the property that the unique limit cycle reached by all its elements depends on μ only and it does not depend on ρ.*

Definition 86 *The regular system $X \subset X_{\Phi}$ is called* ***perfect*** *if $\forall \mu \in \mathbf{B}^n, \forall \rho \in P_n, \forall \rho' \in P_n$,*

$$(\Phi^{\rho}(\cdot)(\mu) \in X \text{ and } \Phi^{\rho'}(\cdot)(\mu) \in X) \Longrightarrow LC_{\mu}^{\rho} = LC_{\mu}^{\rho'}.$$

19 The properties of the regular autonomous systems

Remark 87 *If $X \subset X_{\Phi}$ is regular, then any subsystem $X' \subset X$ is regular and it has the same generator function like X. The intersection and the union of $X \subset X_{\Phi}$ and $X' \subset X_{\Phi}$ are regular: $X \cap X' \subset X_{\Phi}, X \cup X' \subset X_{\Phi}$.*

Theorem 88 *Let be $\Theta_0 \in P^*(\mathbf{B}^n)$. Then $\forall x \in X_{\Phi}^{\Theta_0}, \forall d \in \mathbf{R}$, we have $x \circ \tau^d \in X_{\Phi}^{\Theta_0}$ i.e. $X_{\Phi}^{\Theta_0}$ is invariant to translations.*

Proof. We suppose that $x \in X_{\Phi}^{\Theta_0}$ is given by

$$x(t) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

where $\mu \in \Theta_0$, $\alpha \in \Pi_n$ and $(t_k) \in Seq$ are arbitrary. We take $d \in \mathbf{R}$ arbitrarily and we remark that $(t_k + d) \in Seq$. On the other hand we get

$$(x \circ \tau^d)(t) = x(t - d) =$$

$$= \mu \cdot \chi_{(-\infty,t_0)}(t-d) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t-d) \oplus ...$$
$$... \oplus \Phi^{\alpha^0...\alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t-d) \oplus ...$$
$$= \mu \cdot \chi_{(-\infty,t_0+d)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0+d,t_1+d)}(t) \oplus ...$$
$$... \oplus \Phi^{\alpha^0...\alpha^k}(\mu) \cdot \chi_{[t_k+d,t_{k+1}+d)}(t) \oplus ...$$

thus $x \circ \tau^d \in X_\Phi^{\Theta_0}$. ■

Remark 89 *The dynamical system $\phi = (\mathbf{B}^n, (\Phi^\rho)_{\rho \in P_n})$ is given. The sets $Orb_\rho(\mu)$ with $\rho \in P_n$ fixed do not represent a partition of $\mathbf{B}^n$ when μ runs over $\mathbf{B}^n$ and we give the counterexample represented by $\Phi : \mathbf{B} \to \mathbf{B}, \Phi = 1$ (the constant function), $\rho = \chi_{\{0,1,2,...\}}$ for which*

$$Orb_\rho(0) = \{0, 1\},$$
$$Orb_\rho(1) = \{1\}$$

non-disjoint sets. The consequence is that points from $\mathbf{R} \times \mathbf{B}^n$ exist representing the intersection of several integral curves.

Theorem 90 *Let be $\mu \in \mathbf{B}^n$ and $\rho \in P_n$ arbitrary. Then the line $\mathbf{R} \times \{\mu\}$ is an integral curve $\Longleftrightarrow \forall t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu$.*

Proof. We have

$$\{(t, \Phi^\rho(t)(\mu)) | t \in \mathbf{R}\} = \mathbf{R} \times \{\mu\} \Longleftrightarrow \forall t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu$$

■

20 Accessible states

Definition 91 *We say that the state $\mu' \in \mathbf{B}^n$ is **directly accessible from** $\mu \in \mathbf{B}^n$ if $\mu \neq \mu'$[4] and some $\lambda \in \mathbf{B}^n$ exists with*

$$\mu' = \Phi^\lambda(\mu).$$

Theorem 92 *We take $\Theta_0 \in P^*(\mathbf{B}^n), \mu', \mu'' \in \mathbf{B}^n$ and we suppose that μ'' is directly accessible from μ' :*

$$\exists \lambda \in \mathbf{B}^n, \mu'' = \Phi^\lambda(\mu'). \tag{34}$$

If μ' is accessible from any initial state under the form

$$\forall \mu \in \Theta_0, \exists \rho \in P_n, \exists t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu', \tag{35}$$

then μ'' is accessible from any initial state:

$$\forall \mu \in \Theta_0, \exists \rho' \in P_n, \exists t' > t, \Phi^{\rho'}(t')(\mu) = \mu''. \tag{36}$$

[4]this request is that of a strict accessibility

Proof. Let $\mu \in \Theta_0$ be arbitrary. The truth of (35) for $\rho \in P_n$,

$$\rho = \alpha^0 \cdot \chi_{\{t_0\}} \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}} \oplus ...$$

$\alpha \in \Pi_n$,$(t_k) \in Seq$ and t shows the existence there of two possibilities.

Case $t < t_0$. We choose $\rho' \in P_n$,

$$\rho' = \alpha'^0 \cdot \chi_{\{t'_0\}} \oplus ... \oplus \alpha'^k \cdot \chi_{\{t'_k\}} \oplus ...$$

$\alpha \in \Pi_n$,$(t_k) \in Seq$ such that $\alpha'^0 =$ the λ that makes (34) true and $t' = t'_0 > t$. We have

$$\Phi^{\rho}(t)(\mu) = \mu = \mu',$$

$$\Phi^{\rho'}(t')(\mu) = \Phi^{\rho'}(t'_0)(\mu') = \Phi^{\alpha'^0}(\mu') = \Phi^{\lambda}(\mu') = \mu''.$$

Case $t \geq t_0$ and we can suppose that $k \in \mathbf{N}$ exists such that $t = t_k$. We choose ρ' such that $\alpha'^0 = \alpha^0, ..., \alpha'^k = \alpha^k, \alpha'^{k+1} =$ the λ that makes (34) true, $(t'_k) = (t_k)$ and $t' = t_{k+1}$. We infer

$$\Phi^{\rho'}(t')(\mu) = \Phi^{\rho'}(t_{k+1})(\mu) = \Phi^{\alpha^0 ... \alpha^k \lambda}(\mu) = \Phi^{\lambda}(\Phi^{\alpha^0 ... \alpha^k}(\mu)) =$$

$$= \Phi^{\lambda}(\Phi^{\rho}(t_k)(\mu)) = \Phi^{\lambda}(\Phi^{\rho}(t)(\mu)) = \Phi^{\lambda}(\mu') = \mu''.$$

■

21 Huffman regular autonomous systems

Definition 93 *The autonomous system X is called **Huffman** if it fulfills one of the next two conditions a), b):*

a) $X \in P^*(S^{(n)})$; the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ and the systems $f, g : S^{(n)} \to P^*(S^{(n)})$ exist so that

$$\forall y \in S^{(n)}, \exists \lim_{t \to \infty} \Phi(y(t)) \Longrightarrow \forall x \in f(y), \lim_{t \to \infty} x(t) = \lim_{t \to \infty} \Phi(y(t)), \quad (37)$$

$$\forall x \in S^{(n)}, \exists \lim_{t \to \infty} x(t) \Longrightarrow \forall y \in g(x), \lim_{t \to \infty} y(t) = \lim_{t \to \infty} x(t), \quad (38)$$

$$X = \{x | \exists y \in S^{(n)}, x \in f(y) \text{ and } y \in g(x)\}; \quad (39)$$

b) $X \in P^*(S^{(n+n')})$; the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ and the systems $f, g : S^{(n)} \to P^*(S^{(n)})$, $f' : S^{(n)} \to P^*(S^{(n')})$ exist so that (37), (38) are true as well as

$$X = \{(x, x') | \exists y \in S^{(n)}, x \in f(y), x' \in f'(y) \text{ and } y \in g(x)\}. \quad (40)$$

The two conditions a), b) have been drawn in Figure 5.

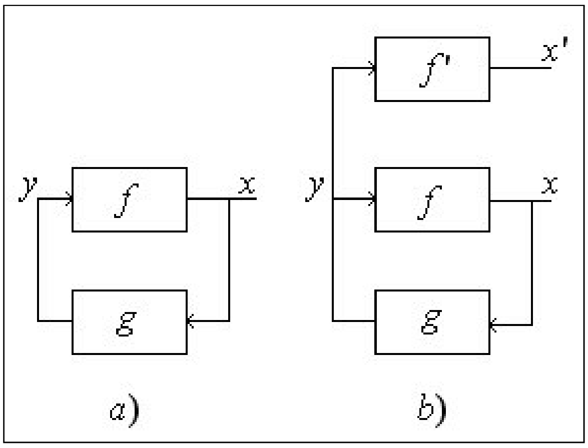

Figure 5:

Remark 94 *A system f having the property that Φ exists with (37) true is called combinational (or race-free stable relative to the function Φ). Property (37) shows that f is a combinational system that computes the function Φ and (38) shows that g is a combinational system that computes the identity function $1_{\mathbf{B}^n}$; (39) links f and g and similarly (40) links f, g, f'. As g from (38) models the delay elements, we conclude that the Huffman autonomous systems consist in combinational systems f having feedback loops with delay elements.*

Theorem 95 *Let be $d > 0$ and the systems $f, g : S^{(n)} \to P^*(S^{(n)}), X_\Phi^d \in P^*(S^{(n)})$ defined in the next manner:*

$$\forall y \in S^{(n)}, f(y) = \{x | x(t) = \begin{cases} \Phi^{\alpha^k}(y(t_k)), t = t_k \\ x(t-0), t \notin \{t_0, ..., t_k, ...\} \end{cases},$$

$$x(-\infty + 0) \in \mathbf{B}^n, \alpha \in \Pi_n, (t_k) \in Seq, \forall k \in \mathbf{N}, t_{k+1} - t_k > d\},$$

$$g(x) = \{x \circ \tau^d\},$$

$$X_\Phi^d = \{x | \exists y \in S^{(n)}, x \in f(y) \text{ and } y \in g(x)\}.$$

Then:

a) f satisfies (37)

b) g satisfies (38)

c) X_Φ^d satisfies (39)

d) $\forall x \in X_\Phi^d$, we have the existence of $\mu \in \mathbf{B}^n, \alpha \in \Pi_n, (t_k) \in Seq$ with $\forall k \in \mathbf{N}, t_{k+1} - t_k > d$ and

$$x(t) = \Phi^{\alpha^0 \cdot \chi_{\{t_0\}} \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}} \oplus ...}(t)(\mu),$$

i.e. $X_\Phi^d \subset X_\Phi$.

Proof. a) Let be $\alpha \in \Pi_n$ and $(t_k) \in Seq$ arbitrary so that $\forall k \in \mathbf{N}, t_{k+1} - t_k > d$. From the hypothesis we can suppose that for an arbitrary $y \in S^{(n)}$, $k' \in \mathbf{N}$ exists so that

$$\forall t \geq t_{k'}, \Phi(y(t)) = \Phi(y(t_{k'})).$$

Define the sets $\Psi_{k'}, ..., \Psi_{k'+p} \subset \{1, ..., n\}, p \in \mathbf{N}$ in the following way:

$$\Psi_{k'} = \{i | i \in \{1, ..., n\}, \alpha_i^{k'} = 1\}, ..., \Psi_{k'+p} = \{i | i \in \{1, ..., n\}, \alpha_i^{k'+p} = 1\},$$

$$\Psi_{k'} \cup ... \cup \Psi_{k'+p} = \{1, ..., n\}.$$

Because α is progressive, the definition of these sets is always possible. We can also suppose that $p \geq 1$. We infer for any $x \in f(y)$ that

$t \in [t_{k'}, t_{k'+1}) : \ \forall i \in \Psi_{k'}$,

$$x_i(t) = \Phi_i(y(t_{k'})),$$

$t \in [t_{k'+1}, t_{k'+2}) : \ \forall i \in \Psi_{k'} \cup \Psi_{k'+1}$,

$$x_i(t) = \begin{cases} \Phi_i(y(t_{k'+1})), i \in \Psi_{k'+1} \\ x_i(t_{k'+1} - 0), i \in \Psi_{k'} \setminus \Psi_{k'+1} \end{cases}$$
$$= \begin{cases} \Phi_i(y(t_{k'})), i \in \Psi_{k'+1} \\ \Phi_i(y(t_{k'})), i \in \Psi_{k'} \setminus \Psi_{k'+1} \end{cases} = \Phi_i(y(t_{k'})),$$

$$...$$

$t \in [t_{k'+p}, \infty) : \ \forall i \in \Psi_{k'} \cup ... \cup \Psi_{k'+p}$,

$$x_i(t) = \begin{cases} \Phi_i(y(t_{k'+p})), i \in \Psi_{k'+p} \\ x_i(t_{k'+p} - 0), i \in (\Psi_{k'} \cup ... \cup \Psi_{k'+p-1}) \setminus \Psi_{k'+p} \end{cases}$$
$$= \begin{cases} \Phi_i(y(t_{k'})), i \in \Psi_{k'+p} \\ \Phi_i(y(t_{k'})), i \in (\Psi_{k'} \cup ... \cup \Psi_{k'+p-1}) \setminus \Psi_{k'+p} \end{cases} = \Phi_i(y(t_{k'})).$$

b) Some $t' \in \mathbf{R}$ exists so that $\forall t \geq t', x(t) = x(t')$, wherefrom

$$\forall t \geq t' + d, y(t) = g(x)(t) = x(t - d) = x(t').$$

c) Obvious.

d) Let be $x \in X_\Phi^d$ arbitrary, in other words $\mu \in \mathbf{B}^n, \alpha \in \Pi_n$ and $(t_k) \in Seq$ exist so that $\forall k \in \mathbf{N}, t_{k+1} - t_k > d$ and

$$\begin{cases} x(-\infty + 0) = \mu \\ x(t) = \begin{cases} \Phi^{\alpha^k}(x(t_k - d)), t = t_k \\ x(t - 0), t \notin \{t_0, ..., t_k, ...\} \end{cases} \end{cases}. \tag{41}$$

We denote

$$\rho(t)=\alpha^0\cdot\chi_{\{t_0\}}(t)\oplus...\oplus\alpha^k\cdot\chi_{\{t_k\}}(t)\oplus...$$

and (41) becomes

$$\begin{cases} x(-\infty+0)=\mu \\ x(t)=\begin{cases}\Phi^{\rho(t_k)}(x(t_k-d)), t=t_k \\ x(t-0), t\notin\{t_0,...,t_k,...\}\end{cases}\end{cases}. \tag{42}$$

Because $x(t)=x(t-0), t\notin\{t_0,...,t_k,...\}$, the only discontinuity points of x (i.e. for which $x(t)\neq x(t-0)$) are found in the set (t_k) thus, taking into account the fact that $t_k-d>t_{k-1}, k\geq 1$ we conclude that

$$\forall k\in\mathbf{N}, x(t_k-d)=x(t_k-0).$$

Equation (42) is equivalent with (i.e. it has the same solution like)

$$\begin{cases} x(-\infty+0)=\mu \\ x(t)=\Phi^{\rho(t)}(x(t-0))\end{cases},$$

see also Theorem 57. ■

22 Delay-insensitivity

Definition 96 *The autonomous system $X\in P^*(S^{(n)})$ is by definition **delay-insensitive** if the following property of stability is true:*

$$\exists\mu'\in\mathbf{B}^n, \forall x\in X, \lim_{t\to\infty}x(t)=\mu'.$$

Theorem 97 *Let be the set $\Theta_0\in P^*(\mathbf{B}^n)$. The next statements concerning the delay-insensitivity of $X_\Phi^{\Theta_0}$ are equivalent:*

$$\exists\mu'\in\mathbf{B}^n, \forall x\in X_\Phi^{\Theta_0}, \lim_{t\to\infty}x(t)=\mu', \tag{43}$$

$$\exists\mu'\in\mathbf{B}^n, \forall\mu\in\Theta_0, \forall\rho\in P_n, \lim_{t\to\infty}\Phi^\rho(t)(\mu)=\mu', \tag{44}$$

$$\exists\mu'\in\mathbf{B}^n, \forall\mu\in\Theta_0, \forall\alpha\in\Pi_n, \lim_{k\to\infty}\Phi^{\alpha^0...\alpha^k}(\mu)=\mu'. \tag{45}$$

Proof. (43)$\Longleftrightarrow$(44) Because $X_\Phi^{\Theta_0}=\{\Phi^\rho(\cdot)(\mu)|\mu\in\Theta_0,\rho\in P_n\}$, the equivalence is obvious.

(44)$\Longrightarrow$(45) Let be $\mu\in\Theta_0, \alpha\in\Pi_n$, $(t_k)\in Seq$ arbitrary and we use the notation

$$\rho(t)=\alpha^0\cdot\chi_{\{t_0\}}(t)\oplus...\oplus\alpha^k\cdot\chi_{\{t_k\}}(t)\oplus... \tag{46}$$

The hypothesis states the existence of $\mu' \in \mathbf{B}^n$ with

$$\exists t' \in \mathbf{R}, \forall t \geq t', \Phi^{\rho}(t)(\mu) = \mu' \tag{47}$$

and we can suppose the existence of some $k' \in \mathbf{N}$ with $t' = t_{k'}$. Because $\forall t \geq t', \exists k \in \mathbf{N}$,

$$t \in [t_{k'+k}, t_{k'+k+1}) \textit{ and } \Phi^{\rho}(t)(\mu) = \Phi^{\alpha^0 ... \alpha^{k'} \alpha^{k'+1} ... \alpha^{k'+k}}(\mu), \tag{48}$$

we have that (47) implies

$$\exists k' \in \mathbf{N}, \forall k \in \mathbf{N}, \Phi^{\alpha^0 ... \alpha^{k'} \alpha^{k'+1} ... \alpha^{k'+k}}(\mu) = \mu'. \tag{49}$$

(45)$\Longrightarrow$(44) Let be $\mu \in \Theta_0, \alpha \in \Pi_n$ and $(t_k) \in Seq$ arbitrary, for which we define ρ like at (46), thus $\rho \in P_n$ is an arbitrary element. The hypothesis states the existence of $\mu' \in \mathbf{B}^n$ such that (49) holds. For $t' = t_{k'}$, because (48) is true, we can see the truth of (47). $\blacksquare$

23 Examples of delay-insensitive systems

Example 98 *The constant function* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$*,*

$$\exists \mu' \in \mathbf{B}^n, \forall \mu \in \mathbf{B}^n, \Phi(\mu) = \mu'$$

fulfills (45) with $\Theta_0 = \mathbf{B}^n$*. Indeed, let* $\mu \in \mathbf{B}^n$ *and* $\alpha \in \Pi_n$ *be arbitrary and we have the following possibilities.*

Case $\mu = \mu'$.

Then $\Phi(\mu) = \mu$ *and (45) is true under the form*

$$\forall k \in \mathbf{N}, \Phi^{\alpha^0 ... \alpha^k}(\mu) = \mu'.$$

Case $\exists p \in \{1, ..., n\}, \exists i_1 \in \{1, ..., n\}, ..., \exists i_p \in \{1, ..., n\}$ *such that* $\mu' = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}$.

We define $q \in \mathbf{N}$ *as the least number that satisfies*

$$\Psi_0 = \{i | i \in \{1, ..., n\}, \alpha_i^0 = 1\}, ..., \Psi_q = \{i | i \in \{1, ..., n\}, \alpha_i^q = 1\},$$

$$\{i_1, ..., i_q\} \subset \Psi_0 \cup ... \cup \Psi_q.$$

We have

$$\Phi^{\alpha^0}(\mu) = \mu \oplus \underset{j \in \Psi_0 \cap \{i_1, ..., i_p\}}{\Xi} \varepsilon^j,$$

$$...$$

$$\Phi^{\alpha^0 ... \alpha^q}(\mu) = \mu \oplus \underset{j \in (\Psi_0 \cup ... \cup \Psi_q) \cap \{i_1, ..., i_p\}}{\Xi} \varepsilon^j = \mu'.$$

From this moment we have for any $q' \geq q$

$$\Phi^{\alpha^0 ... \alpha^{q'}}(\mu) = \mu'.$$

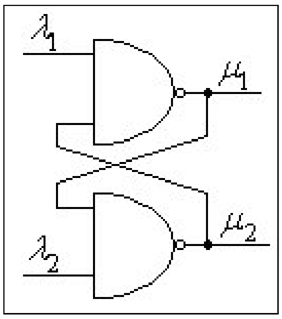

Figure 6:

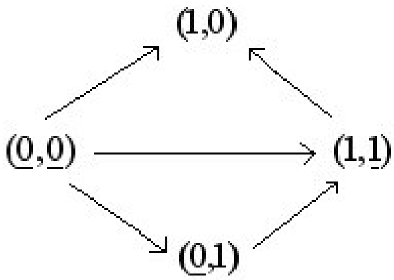

Figure 7:

Notation 99 *We use to underline sometimes the excited coordinates (see Definition 52), for example* $(..., \underline{\mu_i}, ..., \mu_j, ...)$ *shows the fact that*

$$..., \mu_i \neq \Phi_i(\mu), ..., \mu_j = \Phi_j(\mu), ...$$

Notation 100 *If* μ' *is directly accessible from* μ *(see Definition 91), we use to denote this fact by an arrow* $\mu \to \mu'$.

Example 101 *In Figure 6 we have drawn an RS flip-flop where* λ_1, λ_2, μ_1, $\mu_2 \in \mathbf{B}$. *Corresponding to* $\lambda_1 = 0, \lambda_2 = 1$, *respectively to* $\lambda_1 = 1, \lambda_2 = 0$ *we have two functions* $\Phi, \Phi' : \mathbf{B}^2 \to \mathbf{B}^2$,

$$\forall(\mu_1, \mu_2) \in \mathbf{B}^2, \Phi(\mu_1, \mu_2) = (1, \overline{\mu_1}),$$

$$\forall(\mu_1, \mu_2) \in \mathbf{B}^2, \Phi'(\mu_1, \mu_2) = (\overline{\mu_2}, 1)$$

for which (43),...,(45) are fulfilled when $\Theta_0 = \mathbf{B}^2$. *We show the fulfillment of (45) by* Φ *and* $\mu' = (1, 0)$.

The behavior of the circuit from Figure 6 with $\lambda_1 = 0, \lambda_2 = 1$ *is the one from Figure 7.*

Conformably to the Notations 99, 100, the excited coordinates were underlined and the directly accessible states were outlined with arrows. The fact that in the state $(0,0)$ *any coordinate may be computed (both coordinates are exited) implies that we can get from there in any of* $(0,1),(1,1),(1,0)$; *the states* $(0,1)$ *and* $(1,1)$ *have one excited coordinate only, thus one arrow towards some directly accessible state. The only state that has no excited coordinates is* $(1,0)$, *the system arrives and remains there from any initial value* $\mu \in \mathbf{B}^n$.

The satisfaction of (45) is proved.

24 Hazard-freedom, the first definition

Definition 102 *The system* $X \in P^*(S^{(n)})$ *is by definition* ***hazard-free*** *if the property*

$$\exists \mu' \in \mathbf{B}^n, \forall x \in X, x(t) \underset{monotonous}{\rightarrow} \mu'$$

is fulfilled. We have denoted by $\underset{monotonous}{\rightarrow}$ *the coordinatewise monotonous convergence i.e. each coordinate function* x_i *is allowed to change value at most once.*

Remark 103 *The coordinatewise monotonous* $x \in S^{(n)}$ *functions have a final value. The hazard-freedom of* X *states that all the elements* $x \in X$ *are coordinatewise monotonous and that they have the same final value. Note that, like in the case of delay-insensitivity, this definition of hazard-freedom does not ask that* X *is regular.*

Theorem 104 *Let be the set* $\Theta_0 \in P^*(\mathbf{B}^n)$. *The next statements concerning the hazard-freedom of* $X_\Phi^{\Theta_0}$ *are equivalent:*

$$\exists \mu' \in \mathbf{B}^n, \forall x \in X_\Phi^{\Theta_0}, x(t) \underset{monotonous}{\rightarrow} \mu', \tag{50}$$

$$\exists \mu' \in \mathbf{B}^n, \forall \mu \in \Theta_0, \forall \rho \in P_n, \Phi^\rho(t)(\mu) \underset{monotonous}{\rightarrow} \mu', \tag{51}$$

$$\exists \mu' \in \mathbf{B}^n, \forall \mu \in \Theta_0, \forall \alpha \in \Pi_n, \Phi^{\alpha^0 \ldots \alpha^k}(\mu) \underset{monotonous}{\rightarrow} \mu'. \tag{52}$$

Proof. Similar with the proof of Theorem 97. ■

Example 105 *For* $\Theta_0 = \{(0,0,0),(1,1,1)\}$, *we have*

$$(0,0,\underline{0}) \rightarrow (0,\underline{0},1) \rightarrow (0,1,1),$$

$$(\underline{1},1,1) \rightarrow (0,1,1)$$

and because the state $(0,1,1)$ *has no underlined coordinates, it is a fixed point of* Φ *and a final state of the system.*

25 Hazard freedom, the second definition

Remark 106 *In the Definition 102 of hazard-freedom, the vector* $\mu' \in \mathbf{B}^n$ *towards which all* $x \in X$ *converge is independent on* $x(-\infty+0)$. *We give now a possibility of defining a hazard-freedom property for the regular systems where the states* $x \in X$ *converge towards the limit* $\Phi(x(-\infty+0))$.

Definition 107 *The regular system* $X \subset X_\Phi$ *is called* ***hazard-free*** *if the following property is true:*

$$\forall x \in X, x(t) \underset{monotonous}{\rightarrow} \Phi(x(-\infty+0)). \tag{53}$$

Theorem 108 *Let be* $\Theta_0 \in P^*(\mathbf{B}^n)$. *The next statements of hazard-freedom of* $X_\Phi^{\Theta_0}$ *are equivalent:*

$$\forall x \in X_\Phi^{\Theta_0}, x(t) \underset{monotonous}{\rightarrow} \Phi(x(-\infty+0)), \tag{54}$$

$$\forall \mu \in \Theta_0, \forall \rho \in P_n, \Phi^\rho(t)(\mu) \underset{monotonous}{\rightarrow} \Phi(\mu), \tag{55}$$

$$\forall \mu \in \Theta_0, \forall \alpha \in \Pi_n, \Phi^{\alpha^0 \ldots \alpha^k}(\mu) \underset{monotonous}{\rightarrow} \Phi(\mu). \tag{56}$$

Proof. Similar with the proof of Theorem 97. ■

Remark 109 *The theorem which follows gives a property that is equivalent with the hazard-freedom of* $X_\Phi^{\Theta_0}$. *However, at the moment, we do not know the proof of its necessity part.*

Theorem 110 *Let be* $\Theta_0 \in P^*(\mathbf{B}^n)$. *If*

$$\forall \mu \in \Theta_0, \forall \lambda \in \mathbf{B}^n, \Phi(\Phi^\lambda(\mu)) = \Phi(\mu),$$

then $X_\Phi^{\Theta_0}$ *is hazard-free.*

Proof. We show that property (56) is implied and let for this $\mu \in \Theta_0, \alpha \in \Pi_n$ be arbitrary. If $\Phi(\mu) = \mu$ holds, then the conclusion is true thus we suppose the existence of $p \geq 1$ and $\{i_1, ..., i_p\} \subset \{1, ..., n\}$ such that

$$\mu \oplus \Phi(\mu) = \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}.$$

With

$$\Psi_k = \{i | i \in \{1, ..., n\}, \alpha_i^k = 1\}, k \in \mathbf{N},$$

we infer:

$$\Phi^{\alpha^0}(\mu) = \mu \oplus \underset{j \in \Psi_0 \cap \{i_1, ..., i_p\}}{\Xi} \varepsilon^j,$$

$$\Phi^{\alpha^0\alpha^1}(\mu) = \Phi^{\alpha^1}(\Phi^{\alpha^0}(\mu)) = \Phi^{\alpha^1}(\mu \oplus \underset{j\in\Psi_0\cap\{i_1,...,i_p\}}{\Xi} \varepsilon^j) =$$

$$= \begin{cases} \Phi_i(\mu \oplus \underset{j\in\Psi_0\cap\{i_1,...,i_p\}}{\Xi} \varepsilon^j), i \in \Psi_1 \\ \mu_i \oplus \underset{j\in\Psi_0\cap\{i_1,...,i_p\}}{\Xi} \varepsilon_i^j, i \in \{1,...,n\} \setminus \Psi_1 \end{cases}$$

$$= \begin{cases} \Phi_i(\mu), i \in \Psi_1 \\ \mu_i \oplus \underset{j\in\Psi_0\cap\{i_1,...,i_p\}}{\Xi} \varepsilon_i^j, i \in \{1,...,n\} \setminus \Psi_1 \end{cases}$$

$$= \begin{cases} \mu_i \oplus 1, i \in \Psi_1 \cap \{i_1,...,i_p\} \\ \mu_i, i \in \Psi_1 \setminus \{i_1,...,i_p\} \\ \mu_i \oplus 1, i \in (\{1,...,n\} \setminus \Psi_1) \cap \Psi_0 \cap \{i_1,...,i_p\} \\ \mu_i, i \in (\{1,...,n\} \setminus \Psi_1) \setminus (\Psi_0 \cap \{i_1,...,i_p\}) \end{cases}$$

$$= \begin{cases} \mu_i \oplus 1, i \in (\Psi_1 \cap \{i_1,...,i_p\}) \cup ((\{1,...,n\} \setminus \Psi_1) \cap \Psi_0 \cap \{i_1,...,i_p\}) \\ \mu_i, otherwise \end{cases}$$

We denote by $\overline{\Psi_1}$ the set $\{1,...,n\} \setminus \Psi_1$. We compute:

$$(\Psi_1 \cap \{i_1,...,i_p\}) \cup ((\{1,...,n\} \setminus \Psi_1) \cap \Psi_0 \cap \{i_1,...,i_p\}) =$$

$$= ((\Psi_1 \cap \{i_1,...,i_p\}) \cup \overline{\Psi_1}) \cap ((\Psi_1 \cap \{i_1,...,i_p\}) \cup (\Psi_0 \cap \{i_1,...,i_p\}))$$

$$= (\{i_1,...,i_p\} \cup \overline{\Psi_1}) \cap (\Psi_1 \cup \Psi_0) \cap \{i_1,...,i_p\} = (\Psi_1 \cup \Psi_0) \cap \{i_1,...,i_p\}.$$

We can prove by induction on k that the general term is

$$\Phi^{\alpha^0...\alpha^k}(\mu) = \mu \oplus \underset{j\in(\Psi_0\cup...\cup\Psi_k)\cap\{i_1,...,i_p\}}{\Xi} \varepsilon^j$$

and it converges monotonously to

$$\mu \oplus \underset{j\in\{i_1,...,i_p\}}{\Xi} \varepsilon^j = \Phi(\mu).$$

■

Example 111 *Define $\Phi : \mathbf{B}^3 \to \mathbf{B}^3$ by the following table:*

(μ_1, μ_2, μ_3)	Φ
$(0,0,0)$	$(1,1,0)$
$(1,0,0)$	$(1,1,0)$
$(0,1,0)$	$(1,1,0)$
$(1,1,0)$	$(1,1,0)$
$(0,0,1)$	$(1,1,1)$
$(1,0,1)$	$(1,1,1)$
$(0,1,1)$	$(1,1,1)$
$(1,1,1)$	$(1,1,1)$

and we see that for $\Theta_0 = \mathbf{B}^3$ the hazard-freedom property (53) is fulfilled.

Remark 112 *The first definition of hazard-freedom implies delay-insensitivity, but the second doesn't.*

26 Synchronous-likeness

Definition 113 *We define for* $k \in \mathbf{N}$ *the function* $\Phi^{(k)} : \mathbf{B}^n \to \mathbf{B}^n$ *in the next manner:* $\forall \mu \in \mathbf{B}^n$,

$$\Phi^{(0)}(\mu) = \mu,$$

$$\Phi^{(k+1)}(\mu) = \Phi(\Phi^{(k)}(\mu)).$$

$\Phi^{(k)}$ *is called the* ***iterate of order*** k *(or the* $k-$***th iterate****)* ***of*** Φ.

Definition 114 *The autonomous system* $X \subset X_\Phi$ *is* ***synchronous-like*** *if*

$$\forall x \in X, \exists (t_k) \in Seq, \forall k \in \mathbf{N}, x(t_k) = \Phi^{(k+1)}(x(-\infty + 0)). \tag{57}$$

Remark 115 *In formula (57) during some interval* $[t_k, t_{k+1})$ x *may change value. The point is here that the values* $x(-\infty+0)$, $\Phi(x(-\infty+0))$, $\Phi(\Phi(x(-\infty + 0)))$, $\Phi(\Phi(\Phi(x(-\infty + 0))))$, ... *are reached, in this order.*

Theorem 116 *We consider the set* $\Theta_0 \in P^*(\mathbf{B}^n)$. *The next statements concerning the synchronous-likeness of* $X_\Phi^{\Theta_0}$ *are equivalent:*

$$\forall x \in X_\Phi^{\Theta_0}, \exists (t_k) \in Seq, \forall k \in \mathbf{N}, x(t_k) = \Phi^{(k+1)}(x(-\infty + 0)), \tag{58}$$

$$\forall \mu \in \Theta_0, \forall \rho \in P_n, \exists (t_k) \in Seq, \forall k \in \mathbf{N}, \Phi^\rho(t_k)(\mu) = \Phi^{(k+1)}(\mu), \tag{59}$$

$$\forall \mu \in \Theta_0, \forall \alpha \in \Pi_n, \exists (j_k) \in Sub(\mathbf{N}), \forall k \in \mathbf{N}, \tag{60}$$

$$\Phi^{\alpha^0 \ldots \alpha^{j_k}}(\mu) = \Phi^{(k+1)}(\mu).$$

Proof. The line of the proof is similar with the proofs of Theorems 97, 104 and 108. ■

Theorem 117 *Suppose that the system* X *satisfies the hazard-freedom property (53):*

$$\forall x \in X, x(t) \underset{monotonous}{\to} \Phi(x(-\infty + 0)).$$

Then it is synchronous-like.

Proof. For arbitrary $x \in X$, (57) is fulfilled with

$$\forall k \geq 2, \Phi^{(k)}(x(-\infty + 0)) = \Phi(x(-\infty + 0)).$$

■

27 Synchronicity (the technical condition of proper operation)

Definition 118 *The autonomous system $X \subset X_\Phi$ is called **synchronous** and we also say that it fulfills the **technical condition of proper operation** if the next property is satisfied: $\forall x \in X$, $\exists (t_k) \in Seq$,*

$$x(t) = x(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi(x(-\infty + 0)) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \tag{61}$$

$$... \oplus \Phi^{(k+1)}(x(-\infty + 0)) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

Theorem 119 *The set $\Theta_0 \in P^*(\mathbf{B}^n)$ is given. The next statements concerning the synchronicity of $X_\Phi^{\Theta_0}$ are equivalent:*

$$\forall x \in X_\Phi^{\Theta_0}, \exists (t_k) \in Seq, x(t) = x(-\infty + 0) \cdot \chi_{(-\infty,t_0)} \oplus \tag{62}$$

$$\oplus \Phi(x(-\infty + 0)) \cdot \chi_{[t_0,t_1)} \oplus ... \oplus \Phi^{(k)}(\mu) \cdot \chi_{[t_k,t_{k+1})} \oplus ...$$

$$\forall \mu \in \Theta_0, \forall \alpha \in \Pi_n, \exists j_0 \in \mathbf{N}, \exists j_1 \in \mathbf{N}^*, ..., \exists j_k \in \mathbf{N}^*, ... \tag{63}$$

$$(\Phi^{\alpha^0 ... \alpha^k}(\mu))_{k\in\mathbf{N}} = (\underbrace{\mu, ..., \mu}_{j_0}, \underbrace{\Phi(\mu), ..., \Phi(\mu)}_{j_1}, ..., \underbrace{\Phi^{(k)}(\mu), ..., \Phi^{(k)}(\mu)}_{j_k}, ...),$$

where $\underbrace{\mu, ..., \mu}_{0}$ means non-existing values,

$$\forall \mu \in \Theta_0, \forall k \in \mathbf{N}, \Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu) \text{ or} \tag{64}$$

$$\text{or } \exists i \in \{1, ..., n\}, \Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu) \oplus \varepsilon^i.$$

Proof. (62)$\Longrightarrow$(63) Let $\mu \in \Theta_0$, $\alpha \in \Pi_n$ and $(t_k) \in Seq$ be arbitrary, thus

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \tag{65}$$

$$... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

is an arbitrary element from $X_\Phi^{\Theta_0}$. The hypothesis of synchronicity of $X_\Phi^{\Theta_0}$ shows that $\exists (t'_k) \in Seq$ such that

$$x(t) = \mu \cdot \chi_{(-\infty,t'_0)}(t) \oplus \Phi(\mu) \cdot \chi_{[t'_0,t'_1)}(t) \oplus ... \oplus \Phi^{(k)}(\mu) \cdot \chi_{[t'_k,t'_{k+1})}(t) \oplus ... \tag{66}$$

By comparing (65) with (66) the validity of (63) follows.

(63)$\Longrightarrow$(62) We take some arbitrary $\mu \in \Theta_0, \alpha \in \Pi_n, (t_k) \in Seq$ and in this situation (65) represents an arbitrary $x \in X_\Phi^{\Theta_0}$. The hypothesis states the existence of $j_0 \in \mathbf{N}$, $j_1 \in \mathbf{N}^*$, ..., $j_k \in \mathbf{N}^*$, ... such that

$$(\Phi^{\alpha^0 ... \alpha^k}(\mu))_{k\in\mathbf{N}} = (\underbrace{\mu, ..., \mu}_{j_0}, \underbrace{\Phi(\mu), ..., \Phi(\mu)}_{j_1}, ..., \underbrace{\Phi^{(k)}(\mu), ..., \Phi^{(k)}(\mu)}_{j_k}, ...),$$

and this, looking at (65), means the existence of a subsequence $(t'_k) \in Seq$ of (t_k) such that (66) be true. $X_\Phi^{\Theta_0}$ is synchronous.

(63)$\Longrightarrow$(64) We suppose against all reason that (64) is not true and in this situation $\mu \in \Theta_0$, $k \in \mathbf{N}$, $p \in \{2, ..., n\}$ and $i_1, ..., i_p \in \{1, ..., n\}$ distinct exist such that

$$\Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu) \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}.$$

Then $\alpha \in \Pi_n$ and $k' \in \mathbf{N}$ exist such that

$$\Phi^{(k)}(\mu) = \Phi^{\alpha^0 ... \alpha^{k'}}(\mu),$$

$$\Phi^{\alpha^0 ... \alpha^{k'} \alpha^{k'+1}}(\mu) \notin \{\Phi^{(k)}(\mu), \Phi^{(k+1)}(\mu)\}$$

and this happens if we take $\alpha^{k'+1} \in \mathbf{B}^n$ with

$$\exists j \in \{1, ..., p\}, \exists j' \in \{1, ..., p\}, \alpha_j^{k'+1} = 1 \text{ and } \alpha_{j'}^{k'+1} = 0.$$

We have obtained a contradiction with (63).

(64)$\Longrightarrow$(63) Let be $\mu \in \Theta_0$ and $\alpha \in \Pi_n$ arbitrary and we prove (63) by induction on $k \in \mathbf{N}$.

$k = 0 : \Phi^{(0)}(\mu) = \mu$ and we have the possibilities:

- $\Phi^{(1)}(\mu) = \Phi^{(0)}(\mu)$, then $(\Phi^{\alpha^0 ... \alpha^k}(\mu))_{k \in \mathbf{N}}$ has all the terms equal with μ and $j_0 \in \mathbf{N}, j_1 \in \mathbf{N}^*, ..., j_k \in \mathbf{N}^*, ...$ may be taken arbitrarily;

- $\exists i \in \{1, ..., n\}, \Phi^{(1)}(\mu) = \Phi^{(0)}(\mu) \oplus \varepsilon^i$, then

$$\Phi^{\alpha^0}(\mu) = \begin{cases} \mu, \ if \ \alpha_i^0 = 0 \ (\Longrightarrow j_0 \geq 1) \\ \Phi(\mu), \ if \ \alpha_i^0 = 1 \ (\Longrightarrow j_0 = 0) \end{cases}$$

k : from the hypothesis of the induction some $k' \in \mathbf{N}$ exists such that $\Phi^{\alpha^0 ... \alpha^{k'}}(\mu) = \Phi^{(k)}(\mu)$ and we have the possibilities:

- $\Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu)$, then $\Phi^{\alpha^0 ... \alpha^{k'}}(\mu) = \Phi^{\alpha^0 ... \alpha^{k'} \alpha^{k'+1}}(\mu) = ... = \Phi^{(k)}(\mu)$ and $j_k \in \mathbf{N}^*, j_{k+1} \in \mathbf{N}^*, ...$ may be chosen arbitrarily;

- $\exists i \in \{1, ..., n\}, \Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu) \oplus \varepsilon^i$, then

$$\Phi^{\alpha^0 ... \alpha^{k'} \alpha^{k'+1}}(\mu) = \begin{cases} \Phi^{(k)}(\mu), \ if \ \alpha_i^{k'+1} = 0 \\ \Phi^{(k+1)}(\mu), \ if \ \alpha_i^{k'+1} = 1 \end{cases}.$$

■

Remark 120 *The synchronous systems are also synchronous-like. The point is that in (57), during the intervals* $[t_k, t_{k+1})$, *x may change value, while in (61), during the intervals* $[t_k, t_{k+1})$, *x is constant.*

28 The generalized technical condition of proper operation

Definition 121 *The regular system $X \subset X_\Phi$ is said to satisfy the* ***generalized technical condition of proper operation*** *if*

$$\forall x \in X, \forall \nu \in \{\Phi^{(k)}(x(-\infty+0))|k \in \mathbf{N}\}, \forall \lambda \in \mathbf{B}^n,$$

$$\Phi^\lambda(\nu) \neq \Phi(\nu) \Longrightarrow \Phi(\Phi^\lambda(\nu)) = \Phi(\nu).$$

Theorem 122 *If the system X having the generator function Φ satisfies the technical condition of proper operation (synchronicity), then it fulfills also the generalized condition of proper operation.*

Proof. Let be $\lambda \in \mathbf{B}^n$, $\mu \in \mathbf{B}^n$ and $k \in \mathbf{N}$ arbitrary (see (64)), for which $\nu = \Phi^{(k)}(\mu)$.

Case $\Phi(\nu) = \nu$

Then $\Phi^\lambda(\nu) = \nu = \Phi(\nu)$ and the conclusion follows.

Case $\exists i \in \{1, ..., n\}, \Phi(\nu) = \nu \oplus \varepsilon^i$

There are two possibilities, $\Phi^\lambda(\nu) = \nu$ ($\Longrightarrow \Phi(\Phi^\lambda(\mu)) = \Phi(\nu)$), $\Phi^\lambda(\nu) = \Phi(\nu)$ and the conclusion is true in both of them. ■

Theorem 123 *If $X \subset X_\Phi$ satisfies the generalized technical condition of proper operation then it is synchronous-like.*

Proof. Let be $x \in X$ arbitrary,

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0...\alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

with $\mu \in \mathbf{B}^n$, $\alpha \in \Pi_n$ and $(t_k) \in Seq$. If $\Phi(\mu) = \mu$, then $x(t) = \mu$ is the constant function and (57) is fulfilled, thus we can suppose that

$$\exists i_1 \in \{1, ..., n\}, ..., \exists i_p \in \{1, ..., n\}, \Phi(\mu) = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}.$$

We define $q \in \mathbf{N}$ as the least integer such that

$$\Psi_0 = \{i|i \in \{1, ..., n\}, \alpha_i^0 = 1\},$$

$$...$$

$$\Psi_q = \{i|i \in \{1, ..., n\}, \alpha_i^q = 1\},$$

$$\{i_1, ..., i_p\} \subset \Psi_0 \cup ... \cup \Psi_q$$

and we have, see also the proof of Theorem 110

$$\Phi^{\alpha^0...\alpha^q}(\mu) = \mu \oplus \underset{j \in (\Psi_0 \cup ... \cup \Psi_q) \cap \{i_1,...,i_p\}}{\Xi} \varepsilon^j = \Phi(\mu).$$

We can define $t'_0 = t_q$, for which (57) is true under the form $x(t'_0) = \Phi(\mu)$.

The hypothesis of the induction shows the existence of the rank $k_1 \in \mathbf{N}^*$ and of the numbers $t'_0 < t'_1 < ... < t'_{k-1}$ such that $t'_{k-1} = t_{k_1-1}, x(t'_1) = \Phi^{(2)}(\mu), ..., x(t'_{k-1}) = \Phi^{(k)}(\mu)$. If $\Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu)$, then $\forall t \geq t'_{k-1}, x(t) = x(t'_{k-1})$ and (57) is fulfilled with arbitrary $t'_k > t'_{k-1}$ and $x(t'_k) = \Phi^{(k+1)}(\mu)$, thus we can suppose that

$$\exists j_1 \in \{1, ..., n\}, ..., \exists j_{p'} \in \{1, ..., n\}, \Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu) \oplus \varepsilon^{j_1} \oplus ... \oplus \varepsilon^{j_{p'}}.$$

We define $q' \in \mathbf{N}$ as the least integer such that

$$\Psi_{k_1} = \{i | i \in \{1, ..., n\}, \alpha_i^{k_1} = 1\},$$

$$...$$

$$\Psi_{k_1+q'} = \{i | i \in \{1, ..., n\}, \alpha_i^{k_1+q'} = 1\},$$

$$\{j_1, ..., j_{p'}\} \subset \Psi_{k_1} \cup ... \cup \Psi_{k_1+q'}$$

for which we get

$$\Phi^{\alpha^0 ... \alpha^{k_1+q'}}(\mu) = \Phi^{(k)}(\mu) \oplus \underset{j \in (\Psi_{k_1} \cup ... \cup \Psi_{k_1+q'}) \cap \{j_1, ..., j_{p'}\}}{\Xi} \varepsilon^j = \Phi^{(k+1)}(\mu).$$

We define $t'_k = t_{k_1+q'}$ and (57) is fulfilled under the form $x(t'_k) = \Phi^{(k+1)}(\mu)$. ■

29 Discrete time

Notation 124 *Denote by*

$$\mathbf{N}\_ = \{-1, 0, 1, 2, ...\}.$$

Definition 125 *Let be $\alpha \in \Pi_n$. We define Φ* ***at the power*** *$\alpha, \Phi^\alpha : \mathbf{N}\_ \to (\mathbf{B}^n)^{\mathbf{B}^n}$ by*

$$\forall k \in \mathbf{N}\_, \forall \mu \in \mathbf{B}^n, \Phi^\alpha(k)(\mu) = \begin{cases} \mu, k = -1 \\ \Phi^{\alpha^0 ... \alpha^k}(\mu), k \geq 0 \end{cases}.$$

Definition 126 *The couple $\phi' = (\mathbf{B}^n, (\Phi^\alpha)_{\alpha \in \Pi_n})$ is called* ***discrete Boolean dynamical system****. $\mathbf{B}^n$ is the* ***phase space****, or the* ***state space*** *and $\mu \in \mathbf{B}^n$ is called* ***phase****, or* ***state****. The function Φ is the* ***generator function*** *of ϕ' and $\Phi^\alpha, \alpha \in \Pi_n$ are called the* ***computations*** *of Φ. The domain $\mathbf{N}\_$ of the computations Φ^α is the* ***time set*** *and $k \in \mathbf{N}\_$ is the* ***time parameter****.*

Remark 127 *To be compared Definition 125 with Definition 31 and Definition 126 with Definition 33. Many definitions and also the reasoning from this paper may be formulated in discrete time and, as a matter of fact, such discrete time reasoning has been used.*

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